New perturbative method for solving the gravitational N-body problem in general relativity

Slava G. Turyshev[1] and Viktor T. Toth[2]

[1] *Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Drive, Pasadena, CA 91109-0899, USA and*

[2] *Ottawa, Ontario K1N 9H5, Canada*

(Dated: May 1, 2013)

We present a new approach to describe the dynamics of an isolated, gravitationally bound astronomical N-body system in the weak field and slow-motion approximation of general relativity. Celestial bodies are described using an arbitrary energy-momentum tensor and assumed to possess any number of internal multipole moments. The solution of the gravitational field equations in any reference frame is presented as a sum of three terms: i) the inertial flat spacetime in that frame, ii) unperturbed solutions for each body in the system boosted to the coordinates of this frame, and iii) the gravitational interaction term. Such an ansatz allows us to reconstruct all features of the gravitational field and to develop a theory of relativistic reference frames. We use the harmonic gauge conditions to impose a significant constraint on the structure of the post-Galilean coordinate transformation functions that relate global coordinates in the inertial reference frame to the local coordinates of the non-inertial frame associated with a particular body. The remaining parts of these functions are constrained using dynamical conditions, which are obtained by constructing the relativistic proper reference frame associated with a particular body. In this frame, the effect of external forces acting on the body is balanced by the fictitious frame-reaction force that is needed to keep the body at rest with respect to the frame, conserving its relativistic linear momentum. We find that this is sufficient to determine explicitly all the terms of the coordinate transformation. The same method is then used to develop the inverse transformations. The resulting post-Galilean coordinate transformations have an approximate group structure that extends the Poincaré group of global transformations to the case of a gravitational N-body system. We present and discuss the structure of the metric tensors corresponding to the reference frames involved, the rules for transforming relativistic gravitational potentials, the coordinate transformations between frames and the resulting relativistic equations of motion.

PACS numbers: 03.30.+p, 04.25.Nx, 04.80.-y, 06.30.Gv, 95.10.Eg, 95.10.Jk, 95.55.Pe

I. INTRODUCTION

Recent experiments have successfully tested Einstein's general theory of relativity in a variety of ways to remarkable precision [1, 2]. Diverse experimental techniques were used to test relativistic gravity in the solar system, namely: spacecraft Doppler tracking, planetary ranging, lunar laser ranging, dedicated gravity experiments in space and many ground-based efforts [2–4]. Given this phenomenological success, general relativity became the standard theory of gravitation, especially where the needs of astronomy, astrophysics, cosmology and fundamental physics are concerned [2]. The theory is used for many practical purposes involving spacecraft navigation, geodesy and time transfer. It is used to determine the orbits of planets and spacecraft and to describe the propagation of electromagnetic waves in spacetime [2].

As we shall see, finding a solution to the Einstein's equations in the case of an unperturbed one body problem is quite a simple task. However, it turns out that a generalization of the resulting post-Newtonian solution to a system of N extended arbitrary bodies is not straightforward.

A neutral point test particle with no angular momentum follows a geodesic that is completely defined by the external gravitational field. However, the coupling of the intrinsic multipole moments of an extended body to the background gravitational field (present due to external gravitational sources), affects the equations of motion of such a body. Similarly, if a test particle is spinning, its equations of motion must account for the coupling of the body's angular momentum to the external gravitational field. As a result, one must be able to describe the interaction of a body's intrinsic multipole moments and angular momentum with the surrounding gravitational field. Multipole moments are well-defined in the local quasi-inertial reference frame generalizing what was defined for the unperturbed one-body problem. While transforming these quantities from one coordinate frame to another, one should account for the fact that the gravitational interaction is non-linear and, therefore, these moments interact with gravitational fields, affecting the body's motion.

When the Riemannian geometry of the general theory of relativity is concerned, it is well known that coordinate charts are merely labels. Usually, spacetime coordinates have no direct physical meaning and it is essential to

construct the observables as coordinate-independent quantities. Thus, in order to interpret the results of observations or experiments, one picks a specific coordinate system, chosen for the sake of convenience and calculational expediency, formulates a coordinate picture of the measurement procedure, and then one derives the observable out of it. It is also known that an ill-defined reference frame may lead to appearance of non-physical terms that may significantly complicate the interpretation of the data collected [5]. Therefore, in practical problems involving relativistic reference frames, choosing the right coordinate system with clearly understood properties is of paramount importance, even as one recognizes that in principle, all (non-degenerate) coordinate systems are created equal [6].

Before one can solve the *global* problem (the motion of the entire N-body system), the *local* gravitational problem (in the body's vicinity) must be solved first. The correspondence of the *global* and *local* problems is established using coordinate transformations by which representations of physical quantities are transformed from the coordinates of one reference frame to another. It is understood (see [7, 8] for discussion) that, in order to present a complete solution for N-body problem in the general theory of relativity, one must therefore find the solutions to the following three intertwined problems:

1). The *global* problem: We must construct a global inertial frame, such as a barycentric inertial reference frame, for the system under study. This frame can be used to describe the *global* translational motion of N extended bodies comprising the system;

2). The *local* problem: For each body in the system, we must construct a local reference frame. This frame is used to study the gravitational field in the vicinity of a particular body, to establish its internal structure and to determine the features of its rotational motion; and finally,

3). A theory of *coordinate reference frames*: We must establish rules of coordinate transformations between the *global* and *local* frames and use these rules to describe physical processes in either of the two classes of reference frames.

A prerequisite to solving the first two of these problems is knowing the transformation rules between *global* and *local* reference frames. Thus, establishing the theory of astronomical reference frames and solving the equations of motion for celestial bodies become inseparable.

Modern theories of relativistic reference frames [5, 9–15], dealing predominantly with the general theory of relativity, usually take the following approach: As a rule, before solving gravitational field equations, four restrictions (coordinate or gauge conditions) are imposed on the components of the Riemannian metric g_{mn}. These conditions extract a particular subset from an infinite set of spacetime coordinates. Within this subset, the coordinates are linked by smooth differentiable transformations that do not change the coordinate conditions that were imposed. A set of differential coordinate conditions used in leading theories of relativistic reference systems, such as that recommended by the International Astronomical Union (see, for instance [6]), are the harmonic gauge conditions. In addition, a set of specific conditions designed to fix a particular reference frame is added to eliminate most of the remaining degrees of freedom, yielding an explicit form for the coordinate system associated with either frame.

In a recent paper [16], we studied accelerating relativistic reference frames in Minkowski spacetime under the harmonic gauge condition. We showed that the harmonic gauge, which has a very prominent role in gravitational physics starting with the work of Fock [17], allows us to present the accelerated metric in an elegant form that depends only on two harmonic potentials. It also allows reconstruction of a significant part of the structure of the post-Galilean coordinate transformation functions relating inertial and accelerating frames. In fact, using the harmonic gauge, we develop the structure of both the direct and inverse coordinate transformations between inertial and accelerated reference frames. Such a complete set of transformations cannot be found in the current literature, since usually either the direct [6, 11] or the inverse [10] transformation is developed, but not both at the same time. The remaining parts of these functions were determined from dynamical conditions, obtained by constructing the relativistic proper reference frame of an accelerated test particle. In this frame, the effect of external forces acting on the observer are balanced by the fictitious frame-reaction force that is needed to keep the test particle at rest with respect to the frame, conserving its relativistic linear momentum. We find that this is sufficient to determine explicitly all the terms of the coordinate transformation. The same method is then used to develop the inverse transformations. The resulting post-Galilean coordinate transformations exhibit an approximate group structure that extends the Poincaré group of global spacetime transformations to the case of arbitrarily accelerated observers moving in the gravitationbal field.

In the present paper we continue the development of this method and study the dynamics of a gravitationally bounded astronomical N-body system. To constrain the available degrees of freedom, we will again use the harmonic gauge. Using the harmonic gauge condition, we develop the structure of both the direct and inverse harmonic coordinate transformations between global and local reference frames. The method presented in this paper could help to generalize contemporary theories of relativistic reference frames and to allow one to deal naturally with both transformations and the relevant equations of motion in a unified formalism. Finally, the method we present does not rely on a particular theory of gravitation; instead, it uses covariant coordinate transformations to explore the

dynamics in the Minkowski spacetime from a general perspective. Thus, any description from a standpoint of a metric theory of gravity must have our results in the general relativistic limit.

The outline of this paper is as follows:

In Sec. II we introduce notation and briefly study the trivial case of a single unperturbed body, followed by a description of the motion of a system of N weakly interacting, self-gravitating, deformable bodies by generalizing the approximate Lorentz transformations. This is accomplished by introducing acceleration-dependent terms in the coordinate transformations. The resulting transformation is given in a general form that relies on a set of functions that are precisely determined in the subsequent sections.

The local coordinate system of an accelerated observer is not unique. We use the harmonic gauge to constrain the set of co-moving coordinate systems in the accelerated reference frame. We carry out this task and determine the metric tensor describing the accelerated reference frame and the structure of the coordinate transformations that satisfy the harmonic gauge conditions. We observe that the metric in the accelerated frame has an elegant form that depends only on two harmonic potentials, which yield a powerful tool that allows the reconstruction, presented in Sec. III, of a significant part of the structure of the post-Galilean coordinate transformation functions between inertial and accelerating frames.

To fix the remaining degrees of freedom and to specify the proper reference frame of an accelerated observer, we introduce a set of dynamical conditions in Section IV. Specifically, we require that the relativistic linear three-momentum of the accelerated observer in its proper reference frame must be conserved. This conservation introduces the necessary additional constraints to determine uniquely all remaining terms.

A similar approach can also be carried out in reverse, establishing coordinate transformation rules from an accelerating to an inertial reference frame, which is done in Sec. V. Key to the approach presented in this section is the use of the contravariant metric for the accelerating frame, which leads in a straightforward manner to the inverse Jacobian matrix, allowing us to present the inverse transformations in which the roles of the inertial and accelerating coordinates are reversed. Although the calculations are formally very similar to those presented in Secs. II D and IV, subtle differences exist; for this reason, and in order to keep Sec. V as self-contained as possible, we opted in favor of some repetitiveness.

We conclude by discussing these results and presenting our recommendations for future research in Section VI. We also show the correspondence of our results to those obtained previously by other authors.

II. MOTION OF AN ISOLATED N-BODY SYSTEM

Our first goal is to investigate the case of N bodies, forming a gravitationally bound system that is isolated, i.e., free of external gravitational fields, described by a metric that is asymptotically flat. Thus, sufficiently far from the N-body system the structure of its gravitational field should resemble that of the one-body system.

A. The unperturbed one-body system

From a theoretical standpoint, general relativity represents gravitation as a tensor field with universal coupling to the particles and fields of the Standard Model. It describes gravity as a universal deformation of the flat spacetime Minkowski metric, γ_{mn}:

$$g_{mn}(x^k) = \gamma_{mn} + h_{mn}(x^k). \tag{1}$$

Alternatively, it can also be defined as the unique, consistent, local theory of a massless spin-2 field h_{mn}, whose source is the total, conserved energy-momentum tensor (see [18, 19] and references therein).[1]

---

[1] The notational conventions employed here are those used by Landau and Lifshitz [20]: Letters from the second half of the Latin alphabet, $m, n, ... = 0...3$ denote spacetime indices. Greek letters $\alpha, \beta, ... = 1...3$ denote spatial indices. The metric γ_{mn} is that of Minkowski spacetime with $\gamma_{mn} = \mathrm{diag}(+1, -1, -1, -1)$ in the Cartesian representation. The coordinates are formed such that $(ct, \mathbf{r}) = (x^0, x^\alpha)$, where c is the velocity of light. The semicolon $a_{;m} \equiv \nabla_m$ denotes a covariant derivative. We employ the Einstein summation convention with indices being lowered or raised using γ_{mn}. Round brackets surrounding indices denote symmetrization and square brackets denote anti-symmetrization. A dot over any function means a differentiation with respect to time t, defined by $x^0 = ct$. We use negative powers of c as a bookkeeping device for order terms; $g^{[k]}_{mn}, k = 1, 2, 3...$ denotes the k-th term in the series expansion of the metric tensor. Other notations will be explained in the paper.

Classically [21, 22], the general theory of relativity can be defined by postulating the action describing the gravitational field and its coupling to matter fields. Absent a cosmological constant, the propagation and self-interaction of the gravitational field is described by the action

$$\mathcal{S}_{\rm G}[g_{mn}] \;=\; \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}R, \tag{2}$$

where G is Newton's constant, g^{mn} is the inverse of g_{mn}, $g = \det g_{mn}$, R is the trace of the Ricci tensor that in turn is given by $R_{mn} = \partial_k\Gamma^k_{mn} - \partial_m\Gamma^k_{nk} + \Gamma^k_{mn}\Gamma^l_{kl} - \Gamma^k_{ml}\Gamma^l_{nk}$, and $\Gamma^k_{mn} = \frac{1}{2}g^{kp}(\partial_m g_{pn} + \partial_n g_{pm} - \partial_p g_{mn})$ are the Christoffel symbols.

The coupling of g_{mn} to all matter fields (this would generally mean all the fields of the Standard Model of particle physics) is accomplished by using it to replace the Minkowski metric everywhere [2]. Varying the total action

$$\mathcal{S}_{\rm tot}[\psi, A_m, H; g_{mn}] = \mathcal{S}_{\rm G}[g_{mn}] + \mathcal{S}_{\rm SM}[\psi, A_m, H; g_{mn}], \tag{3}$$

with respect to g_{mn} we obtain Einstein's field equations for gravity:

$$R^{mn} = \frac{8\pi G}{c^4}\Big(T^{mn} - \frac{1}{2}g^{mn}T\Big), \tag{4}$$

where $T^{mn} = (2/\sqrt{-g})\delta\mathcal{L}_{\rm SM}/\delta g_{mn}$ is the (presumed to by symmetric) energy-momentum tensor of matter, represented by the Lagrangian density $\mathcal{L}_{\rm SM}$. As the density of the Einstein's tensor is conserved, namely $\nabla_k[\sqrt{-g}(R^{mk} - \frac{1}{2}g^{mk}R)] = 0$, it follows that the density of the energy-momentum tensor of matter $\sqrt{-g}T^{mn}$ is also conserved, obeying the following covariant conservation equation:

$$\nabla_k\big(\sqrt{-g}T^{mk}\big) = 0. \tag{5}$$

Einstein's equations (4) connect the geometry of a four-dimensional Riemannian manifold representing spacetime to the stress-energy-momentum of matter contained in that spacetime. The theory is invariant under arbitrary coordinate transformations: $x'^m = f^m(x^n)$. This freedom to choose coordinates allows us to introduce gauge conditions that may help with solving the field equations (4). For instance, in analogy with the Lorenz gauge ($\partial_m A^m = 0$) of electromagnetism, the harmonic gauge corresponds to imposing the condition [17]:

$$\partial_n\big(\sqrt{-g}g^{mn}\big) = 0. \tag{6}$$

We can now proceed with finding a solution to the Einstein's equations (4) that satisfy the harmonic gauge (6).

B. Solution to the unperturbed one-body problem

Equations (4) represent a set of non-linear hyperbolic-type differential equations of the second order with respect to the metric tensor of the Riemannian spacetime. This non-linearity makes finding a solution to this set of equations in the general case to be a complicated problem to which no analytical solution is known and a full numerical treatment is needed. Depending on a particular situation one usually introduces relevant small parameters and develops a solution iteratively.

When studying a problem in the weak gravitational field and slow motion approximation, one often uses the ratio v/c of the velocity of motion of the bodies v with respect to barycentric coordinate reference frame to the speed of light c. For the bodies of the solar system this ratio is a small parameter and typically is of the order $v/c \approx 10^{-4}$. In our formulation, we use the dimensioned parameter c^{-1} as a bookkeeping device for these order terms. Since we are considering a gravitationally bound N-body system in the weak field and slow motion approximation, there exist relations linked by the virial theorem: $v^2/c^2 \sim GM/(c^2R)$ [17, 23], with M, R, and v being the mass, radius and escape velocity of the body. Thus, when we write $y_0 + c^{-2}\mathcal{K}$, for instance, this implies that $\mathcal{K}$ is of order v^2y^0 with $v \ll c$ being the velocity of an accelerating observer or test particle under consideration and $c^{-2}\mathcal{K}$ is of order $(v/c)^2y^0$, which remains small relative to y^0.

To find the solution of the gravitational field equations in first post-Newtonian approximation of the general theory of relativity, we expand the metric tensor g_{mn} (1):

$$g_{00} \;=\; 1 + c^{-2}g^{[2]}_{00} + c^{-4}g^{[4]}_{00} + \mathcal{O}(c^{-6}), \tag{7}$$

$$g_{0\alpha} \;=\; c^{-3}g^{[3]}_{0\alpha} + \mathcal{O}(c^{-5}), \tag{8}$$

$$g_{\alpha\beta} = \gamma_{\alpha\beta} + c^{-2} g^{[2]}_{\alpha\beta} + \mathcal{O}(c^{-4}), \tag{9}$$

where $\gamma_{\alpha\beta} = (-1,-1,-1)$ is the spatial part of the Minkowski metric γ_{mn}. Note that $g_{0\alpha}$ starts with a term of order c^{-3}, which is the lowest order gravitational contribution to this component.

In our calculations we constrain the remaining coordinate freedom in the field equations by imposing the covariant harmonic gauge conditions in the following form of Eq. (6). Thus, for $n=0$ and $n=\alpha$, we correspondingly have:

$$\frac{1}{2}\partial_0\Big(\gamma^{\epsilon\nu} g^{[2]}_{\epsilon\nu} - g^{[2]}_{00}\Big) - \partial^\nu g^{[3]}_{0\nu} = \mathcal{O}(c^{-5}), \tag{10}$$

$$\frac{1}{2}\partial^\alpha\Big(g^{[2]}_{00} + \gamma^{\epsilon\nu} g^{[2]}_{\epsilon\nu}\Big) - \gamma^{\alpha\mu}\partial^\nu g^{[2]}_{\mu\nu} = \mathcal{O}(c^{-4}). \tag{11}$$

The metric tensor given by Eqs. (7)–(9) and the gauge conditions given by Eqs. (10)–(11) allow one to simplify the expressions for the Ricci tensor and to present its contravariant components $R^{mn} = g^{mk} g^{nl} R_{kl}$ in the following form:

$$R^{00} = \frac{1}{2}\Box\Big(c^{-2} g^{[2]00} + c^{-4}\Big\{g^{[4]00} - \tfrac{1}{2}\big(g^{[2]00}\big)^2\Big\}\Big) + c^{-4}\partial_\epsilon\partial_\lambda g^{[2]00}\Big(g^{[2]\epsilon\lambda} + \gamma^{\epsilon\lambda} g^{[2]00}\Big) + \mathcal{O}(c^{-6}), \tag{12}$$

$$R^{0\alpha} = c^{-3}\frac{1}{2}\Delta g^{[3]0\alpha} + \mathcal{O}(c^{-5}), \tag{13}$$

$$R^{\alpha\beta} = c^{-2}\frac{1}{2}\Delta g^{[2]\alpha\beta} + \mathcal{O}(c^{-4}), \tag{14}$$

where $\Box = \gamma^{mn}\partial_m\partial_n = \partial^2_{00} + \partial_\epsilon\partial^\epsilon$ and $\Delta = \partial_\epsilon\partial^\epsilon$ are the d'Alembert and Laplace operators of the Minkowski spacetime correspondingly.

To solve the gravitational field equations (4), we need to specify the form of the energy-momentum tensor. However, it is sufficient to make only some very general assumptions on the form of this tensor that are valid within the post-Newtonian approximation. Indeed, we will only assume that the components of the energy-momentum tensor, T^{mn}, have the following form:

$$T^{00} = c^2\Big(T^{[0]00} + c^{-2} T^{[2]00} + \mathcal{O}(c^{-4})\Big), \quad T^{0\alpha} = c\Big(T^{[1]0\alpha} + \mathcal{O}(c^{-2})\Big), \quad T^{\alpha\beta} = T^{[2]\alpha\beta} + \mathcal{O}(c^{-2}). \tag{15}$$

This form is sufficient to construct an iterative solution scheme in the post-Newtonian approximation of the general theory of relativity leaving the precise form of the energy-momentum tensor T^{mn} unspecified. These assumptions allow us to present the "source term", $S^{mn} = T^{mn} - \frac{1}{2} g^{mn} T$, on the right-hand side of Eq. (4) in the following form:

$$S^{00} = \frac{1}{2}c^2\Big(\sigma + \mathcal{O}(c^{-4})\Big), \tag{16}$$

$$S^{0\alpha} = c\Big(\sigma^\alpha - c^{-2}\frac{1}{2}\sigma g^{[3]0\alpha} + \mathcal{O}(c^{-4})\Big), \tag{17}$$

$$S^{\alpha\beta} = -\gamma^{\alpha\beta}\frac{1}{2}c^2\sigma\big(1 + c^{-2} 2 g^{[2]}_{00}\big) + \sigma^{\alpha\beta} + \mathcal{O}(c^{-2}). \tag{18}$$

where we have introduced the scalar, vector, and shear densities σ, σ^α, and $\sigma^{\alpha\beta}$:

$$\sigma = T^{[0]00} + c^{-2}\big(T^{[2]00} - \gamma_{\mu\lambda} T^{[2]\mu\lambda}\big) + \mathcal{O}(c^{-4}) \equiv \frac{1}{c^2}\Big(T^{00} - \gamma_{\mu\lambda} T^{\mu\lambda}\Big) + \mathcal{O}(c^{-4}), \tag{19}$$

$$\sigma^\alpha = T^{[1]0\alpha} + c^{-2} T^{[3]0\alpha} + \mathcal{O}(c^{-4}) \equiv \frac{1}{c} T^{0\alpha} + \mathcal{O}(c^{-4}), \tag{20}$$

$$\sigma^{\alpha\beta} = T^{[2]\alpha\beta} - \gamma^{\alpha\beta}\gamma_{\mu\lambda} T^{[2]\mu\lambda} + \mathcal{O}(c^{-2}) \equiv T^{\alpha\beta} - \gamma^{\alpha\beta}\gamma_{\mu\lambda} T^{\mu\lambda} + \mathcal{O}(c^{-2}). \tag{21}$$

We can now express the stress-energy tensor T^{mn} in terms of σ^{mn} introduced by Eq. (19)–(21):

$$T^{00} = c^2\sigma - \frac{1}{2}\gamma_{\mu\lambda}\sigma^{\mu\lambda} + \mathcal{O}(c^{-2}), \qquad T^{0\alpha} = c\Big(\sigma^\alpha + \mathcal{O}(c^{-4})\Big), \qquad T^{\alpha\beta} = \sigma^{\alpha\beta} - \gamma^{\alpha\beta}\frac{1}{2}\gamma_{\mu\lambda}\sigma^{\mu\lambda} + \mathcal{O}(c^{-2}). \tag{22}$$

Substituting the expressions (12)–(14) for the Ricci tensor together with the source term S^{mn} given by Eqs. (16)–(21) into the field equations (4) of the general theory of relativity, we obtain the following equations to determine the components of the metric tensor in the post-Newtonian approximation:

$$\Box\Big\{\frac{1}{c^2} g^{[2]00} + \frac{1}{c^4}\Big(g^{[4]00} - \tfrac{1}{2}\big(g^{[2]00}\big)^2\Big)\Big\} + \frac{1}{c^4}\partial_\epsilon\partial_\lambda g^{[2]00}\Big(g^{[2]\epsilon\lambda} + \gamma^{\epsilon\lambda} g^{[2]00}\Big) = \frac{8\pi G}{c^2}\sigma + \mathcal{O}(c^{-6}), \tag{23}$$

$$\frac{1}{c^3}\Delta g^{[3]0\alpha} = \frac{16\pi G}{c^3}\sigma^\alpha + \mathcal{O}(c^{-5}), \tag{24}$$
$$\frac{1}{c^2}\Delta g^{[2]\alpha\beta} = -\gamma^{\alpha\beta}\frac{8\pi G}{c^2}\sigma + \mathcal{O}(c^{-4}). \tag{25}$$

We assume that spacetime is asymptotically flat (no external gravitational field far from the body) and there is no gravitational radiation coming from outside the body. In terms of perturbations of the the Minkowski metric $h_{mn} = g_{mn} - \gamma_{mn}$ introduced by Eq. (1), the corresponding two boundary conditions [17] have the form

$$\lim_{\substack{r\to\infty\\ t+r/c=\mathrm{const}}} h_{mn} = 0 \qquad \text{and} \qquad \lim_{\substack{r\to\infty\\ t+r/c=\mathrm{const}}} \big[(rh_{mn})_{,r} + (rh_{mn})_{,0}\big] = 0. \tag{26}$$

Multiplying Eq. (23) by $\gamma^{\alpha\beta}$ and summing the result with Eq. (25) allows one to determine $g^{[2]\alpha\beta}+\gamma^{\alpha\beta}g^{[2]00} = \mathcal{O}(c^{-2})$, which was derived using the boundary conditions (26). As a result, the system of equations (23)–(25) can equivalently be re-written in the following form:

$$\Box\Big(\frac{1}{c^2}g^{[2]00} + \frac{1}{c^4}\Big\{g^{[4]00} - \tfrac{1}{2}\big(g^{[2]00}\big)^2\Big\}\Big) = \frac{8\pi G}{c^2}\sigma + \mathcal{O}(c^{-6}), \tag{27}$$
$$\frac{1}{c^3}\Delta g^{[3]0\alpha} = \frac{16\pi G}{c^3}\sigma^\alpha + \mathcal{O}(c^{-5}), \tag{28}$$
$$g^{[2]\alpha\beta} + \gamma^{\alpha\beta}g^{[2]00} = \mathcal{O}(c^{-2}). \tag{29}$$

Solution to this system of equations may be given as below:

$$g^{00} = 1 + \frac{2w}{c^2} + \frac{2w^2}{c^2} + \mathcal{O}(c^{-6}), \qquad g^{0\alpha} = \frac{4w^\alpha}{c^3} + \mathcal{O}(c^{-5}), \qquad g^{\alpha\beta} = \gamma^{\alpha\beta} - \gamma^{\alpha\beta}\frac{2w}{c^2} + \mathcal{O}(c^{-4}), \tag{30}$$

where the scalar and vector gravitational potentials w and w^α are determined from the following harmonic equations:

$$\Box w = 4\pi G\sigma + \mathcal{O}(c^{-4}), \qquad \Delta w^\alpha = 4\pi G\sigma^\alpha + \mathcal{O}(c^{-2}), \tag{31}$$

which, according to the harmonic gauge conditions (10)–(11) and the energy-momentum conservation equation (5), satisfy the following two Newtonian continuity equations

$$c\partial_0 w + \partial_\epsilon w^\epsilon = \mathcal{O}(c^{-2}) \quad \text{and} \quad c\partial_0\sigma + \partial_\epsilon\sigma^\epsilon = \mathcal{O}(c^{-2}). \tag{32}$$

Assuming that spacetime asymptotically flat (26), one can write a solution for w and w^α in terms of the advanced and retarded potentials. The recommended solution [6], half advanced and half retarded, reads

$$w(t,\mathbf{x}) = G\int d^3x'\frac{\sigma(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{1}{2c^2}G\frac{\partial^2}{\partial t^2}\int d^3x'\sigma(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'| + \mathcal{O}(c^{-3}), \tag{33}$$
$$w^\alpha(t,\mathbf{x}) = G\int d^3x'\frac{\sigma^\alpha(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \mathcal{O}(c^{-2}). \tag{34}$$

Finally, we can present the unperturbed solution for an isolated one-body problem in terms of Minkowski metric perturbations h_{mn} introduced by Eq. (1) as below:

$$g_{mn} = \gamma_{mn} + h^{(0)}_{mn},$$
$$h^{(0)}_{00} = -\frac{2w}{c^2} + \frac{2w^2}{c^2} + \mathcal{O}(c^{-6}), \qquad h^{(0)}_{0\alpha} = -\gamma_{\alpha\lambda}\frac{4w^\lambda}{c^3} + \mathcal{O}(c^{-5}), \qquad h^{(0)}_{\alpha\beta} = \gamma_{\alpha\beta}\frac{2w}{c^2} + \mathcal{O}(c^{-4}). \tag{35}$$

Equations (35) represent the well-known solution for the one-body problem in the general theory of relativity [1, 2, 18]. The new method, presented here, relies on the properties of this solution in developing a perturbation theory needed to find a solution in the case of the N-body problem.

C. Ansatz for the N-body problem

To describe the dynamics of the N-body problem we introduce $N+1$ reference frames, each with its own coordinate chart. We need a *global* coordinate chart $\{x^m\}$ defined for an inertial reference frame that covers the entire system

under consideration. In the immediate vicinity of each body b in the N-body system, we can also introduce a chart of *local* coordinates $\{y_b^m\}$ defined in the frame associated with this body. In the remainder of this paper, we use $\{x^m\}$ to represent the coordinates of the global inertial frame and $\{y_b^m\}$ to be the local coordinates of the frame associated with a particular body b.

First, we assume that the studied N-body system is isolated and, similarly to the one body problem, there exists a barycentric inertial system (with global coordinates denoted as $x^k \equiv (ct, \mathbf{x})$) that asymptotically resembles the properties of the one-body problem discussed above, namely the metric tensor, $g_{mn}(x^k)$, describing the N-body system in that reference frame is asymptotically flat (no gravitational fields far from the system) and there is no gravitational radiation coming from outside the system.

Therefore, in terms of the perturbations of the Minkowski metric $h^{mn} = g^{mn} - \gamma^{mn}$, the boundary conditions are identical to those given by (26) and, in the case of the N-body problem can be given as below:

$$\lim_{\substack{r\to\infty \\ t+r/c=\mathrm{const}}} h^{mn} = 0 \qquad \text{and} \qquad \lim_{\substack{r\to\infty \\ t+r/c=\mathrm{const}}} \Big[(rh^{mn})_{,r} + (rh^{mn})_{,0}\Big] = 0. \tag{36}$$

Second, we assume that there exists a local reference frame (with coordinates denoted as $y_b^k \equiv (ct_b, \mathbf{y}_b)$) associated with each body b in the N-body system. We further assume that there exists a smooth transformation connecting coordinates y_b^k chosen in this local reference frame to coordinates x^k in the inertial frame. In other words, we assume that there exist both direct $x^k = g_b^k(y_b^l)$ and inverse $y_b^k = f_b^k(x^l)$ coordinate transformation functions between the two reference frames. Our objective is to construct these transformations in explicit form.

The most general form of the post-Galilean coordinate transformations between the non-rotating reference frames defined by global coordinates $x^k \equiv (x^0, x^\alpha)$, and local coordinates $y_a^k \equiv (y_a^0, y_a^\alpha)$ may be given in the following form [8, 16]:

$$x^0 = y_a^0 + c^{-2}\mathcal{K}_a(y_a^0, y_a^\epsilon) + c^{-4}\mathcal{L}_a(y_a^0, y_a^\epsilon) + \mathcal{O}(c^{-6}), \tag{37}$$

$$x^\alpha = y_a^\alpha + z_{a_0}^\alpha(y_a^0) + c^{-2}\mathcal{Q}_a^\alpha(y_a^0, y_a^\epsilon) + \mathcal{O}(c^{-4}), \tag{38}$$

where $z_{a_0}^\mu$ is the Galilean vector connecting the spatial origins of two non-rotating the frames, and we introduce the post-Galilean vector $x_{a_0}^\mu(y^0)$ connecting the origins of the two frames

$$x_{a_0}^\mu(y_a^0) = z_{a_0}^\mu + c^{-2}\mathcal{Q}_a^\mu(y_a^0, 0) + \mathcal{O}(c^{-4}). \tag{39}$$

The functions $\mathcal{K}_a, \mathcal{L}_a$ and $\mathcal{Q}_a^\mu$ are yet to be determined.

The corresponding inverse transformations (further discussed in Sec. V) are given by

$$y_a^0 = x^0 + c^{-2}\hat{\mathcal{K}}_a(x^0, x^\epsilon) + c^{-4}\hat{\mathcal{L}}_a(x^0, x^\epsilon) + \mathcal{O}(c^{-6}), \tag{40}$$

$$y_a^\alpha = x^\alpha - z_{a_0}^\alpha(x^0) + c^{-2}\hat{\mathcal{Q}}_a^\alpha(x^0, x^\epsilon) + \mathcal{O}(c^{-4}), \tag{41}$$

where $z_{a_0}^\mu(x^0)$ is the Galilean position vector of the body a, expressed as a function of global time, x^0.

The local coordinates y_a are expected to remain accurate in the neighborhood of the world-line of the body being considered. The functions $\mathcal{K}_a, \mathcal{L}_a$ and $\mathcal{Q}_a^\mu$ should contain the information about the specific physical properties of the reference frame chosen for the analysis and must depend only on the mutual dynamics between the reference frames, i.e., velocity, acceleration, etc. Furthermore, we would like these transformations to be smooth, which would warrant the existence of inverse transformations discussed in Sec. V.

The coordinate transformation rules for the general coordinate transformations above are easy to obtain and express in the form of the Jacobian matrix $\partial x^m/\partial y^n$. Using Eqs. (37)–(38), we get:

$$\frac{\partial x^0}{\partial y_a^0} = 1 + c^{-2}\frac{\partial \mathcal{K}_a}{\partial y_a^0} + c^{-4}\frac{\partial \mathcal{L}_a}{\partial y_a^0} + \mathcal{O}(c^{-6}), \qquad \frac{\partial x^0}{\partial y_a^\alpha} = c^{-2}\frac{\partial \mathcal{K}_a}{\partial y_a^\alpha} + c^{-4}\frac{\partial \mathcal{L}_a}{\partial y_a^\alpha} + \mathcal{O}(c^{-5}), \tag{42}$$

$$\frac{\partial x^\alpha}{\partial y_a^0} = \frac{v_{a_0}^\alpha}{c} + c^{-2}\frac{\partial \mathcal{Q}_a^\alpha}{\partial y_a^0} + \mathcal{O}(c^{-5}), \qquad \frac{\partial x^\alpha}{\partial y_a^\mu} = \delta_\mu^\alpha + c^{-2}\frac{\partial \mathcal{Q}_a^\alpha}{\partial y_a^\mu} + \mathcal{O}(c^{-4}), \tag{43}$$

where $v_{0a}^\epsilon = \dot{z}_{0a}^\epsilon \equiv cdz_{0a}^\epsilon/dy_a^0$ is the time-dependent velocity of the frame a relative to the barycentric inertial frame.

We can verify that, in order for Eqs. (40)–(41) to be inverse to Eqs. (37)–(38), the "hatted" functions $(\hat{\mathcal{K}}_a, \hat{\mathcal{L}}_a, \hat{\mathcal{Q}}_a^\alpha)$ must relate to the original set of $(\mathcal{K}_a, \mathcal{L}_a, \mathcal{Q}_a^\alpha)$ via the following expressions:

$$\hat{\mathcal{K}}_a(x^k) = -\mathcal{K}_a(x^0, r_a^\epsilon) + \mathcal{O}(c^{-4}), \tag{44}$$

$$\hat{\mathcal{Q}}_a^\alpha(x^k) = (v_{a_0}^\alpha/c)\,\mathcal{K}_a(x^0, r_a^\epsilon) - \mathcal{Q}_a^\alpha(x^0, r_a^\epsilon) + \mathcal{O}(c^{-2}), \tag{45}$$

$$\hat{\mathcal{L}}_a(x^k) \;=\; \frac{\partial\mathcal{K}_a(x^0,r^\epsilon_a)}{\partial x^0}\mathcal{K}_a(x^0,r^\epsilon_a)+\frac{\partial\mathcal{K}_a(x^0,r^\epsilon_a)}{\partial x^\lambda}\mathcal{Q}^\lambda_a(x^0,r^\epsilon_a)-\mathcal{L}_a(x^0,r^\epsilon_a)+\mathcal{O}(c^{-2}), \tag{46}$$

with $r^\epsilon_a=x^\epsilon-x^\epsilon_{a_0}$, where $x^\mu_{a_0}(x^0)$ is the post-Galilean position vector of the body a expressed as a function of the global time-like coordinate x^0 (as opposed to Eq. (39), which is given in local time y^0_a) defined as

$$x^\mu_{a_0}=z^\mu_{a_0}-c^{-2}\hat{\mathcal{Q}}^\mu_a(x^0,0)+\mathcal{O}(c^{-4}). \tag{47}$$

Also, $v^\alpha_{a_0}=\dot{z}^\alpha_{a_0}$ and $x^k\equiv(x^0,x^\epsilon)$.

The transformations given by Eqs. (37)–(38) transform space and time coordinates from the global, inertial frame $\{x^m\}$ to space and time coordinates in the local, accelerating reference frame $\{y^m_a\}$. These transformations are not singular in those regions where the post-Galilean approximation remains valid. To check this, we can verify that the determinant of the Jacobian matrix does not vanish [10]:

$$\det\Big(\frac{\partial x^k}{\partial y^m_a}\Big)=1+c^{-2}\Big\{\frac{\partial\mathcal{K}_a}{\partial y^0_a}-v^\lambda_{a_0}\frac{\partial\mathcal{K}_a}{\partial y^\lambda_a}+\frac{\partial\mathcal{Q}^\mu_a}{\partial y^\mu_a}\Big\}+\mathcal{O}(c^{-4}). \tag{48}$$

This guarantees the invertibility of the Jacobian matrix and hence, the existence of the inverse transformations. The inverse of the Jacobian matrix (42)–(43), $\partial y^n/\partial x^m$, can be obtained directly from (40)–(41):

$$\frac{\partial y^0_a}{\partial x^0} \;=\; 1+c^{-2}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}+c^{-4}\frac{\partial\hat{\mathcal{L}}_a}{\partial x^0}+\mathcal{O}(c^{-6}), \qquad \frac{\partial y^0_a}{\partial x^\alpha}=c^{-2}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^\alpha}+c^{-4}\frac{\partial\hat{\mathcal{L}}_a}{\partial x^\alpha}+\mathcal{O}(c^{-5}), \tag{49}$$

$$\frac{\partial y^\alpha_a}{\partial x^0} \;=\; -\frac{v^\alpha_{a_0}}{c}+c^{-2}\frac{\partial\hat{\mathcal{Q}}^\alpha_a}{\partial x^0}+\mathcal{O}(c^{-5}), \qquad \frac{\partial y^\alpha_a}{\partial x^\mu}=\delta^\alpha_\mu+c^{-2}\frac{\partial\hat{\mathcal{Q}}^\alpha_a}{\partial x^\mu}+\mathcal{O}(c^{-4}). \tag{50}$$

Third, we will search for the solution of the gravitational field equations, g^{mn}, in the case of N-body problem within the general theory of relativity in barycentric coordinate frame in the following form:

$$g^{mn}(x)=\gamma^{mn}(x)+\sum_{b=1}^N\frac{\partial x^m}{\partial y^k_b}\frac{\partial x^n}{\partial y^l_b}\;h^{kl}_{b(0)}(y_b(x))+h^{mn}_{\rm int}(x) \tag{51}$$

where the terms on the right-hand side have the following meaning:

- The first term, $\gamma^{mn}(x)=(1,-1,-1,-1)$, is the contravariant form of the Minkowski metric of flat spacetime;
- The second term is a summation term representing the superposition of the unperturbed one-body solutions $h^{kl}_{b(0)}$ for each body b in the system of equations (35), which were boosted to the barycentric frame by coordinate transformations (37)–(38);
- Finally, the last term, $h^{mn}_{\rm int}$, represents the gravitational interaction between the bodies in the N-body system. It is expected that the interaction term is at least of the second order in the gravitational constant, $h^{mn}_{\rm int}\propto G^2\approx c^{-4}$; this assumption will be used in our approximation method.

Note that the contravariant components of the metric tensor g^{mn} allow us to rely directly on the form of the coordinate transformation $x^m=f^m(y^k_a)$ from Eqs. (37)–(38) and the associated Jacobian matrix $\partial x^m/\partial y^n_a$ given by Eqs. (42)–(43). Because of the metric nature of the general theory of relativity, Eq. (51) is valid for any admissible covariant coordinate transformations $x^m=f^m(y^k_b)$. As, at this stage we already know the unperturbed one-body solution $h^{kl}_{b(0)}$, which is given for each body b in the form of Eqs. (35), the only unknown part of the ansatz is the gravitational interaction term, $h^{mn}_{\rm int}$, which will be determined in Sec. II F.

The form of the iterative solution for N-body metric given by Eq. (51) was inspired by a successful perturbation theory approach of high-energy physics. In that approach, a solution for a system of interacting fields is sought in the form of a sum of the unperturbed solutions for each of the field in the system, plus the interaction term. The fact that the line element, formed using the metric tensor, represents a Lagrangian for test particles (see discussion in Sec. IV A), allows one to develop an iterative solution in the case of N interacting bodies based on the principles of the perturbation theory approach.

Fourth, we impose the harmonic gauge conditions given by Eq. (6), $\partial_n\big(\sqrt{-g}g^{mn}\big)=0$, on the N-body metric tensor g^{mn} given by Eq. (51), in both the global and local frames. As we will see in the following section, this condition that is imposed on the metric tensor leads to the requirement that the transformation functions (37)–(38) must be harmonic. Below we investigate the impact of the harmonic gauge on the transformation functions $\mathcal{K}_a,\mathcal{L}_a$ and $\mathcal{Q}^\mu_a$ and the inverse transformation functions $\hat{\mathcal{K}}_a$, $\hat{\mathcal{Q}}_a$ and $\hat{\mathcal{L}}_a$.

D. Imposing the harmonic gauge conditions

The dynamical condition, i.e., the requirement that the spatial origin of the transformed system of coordinates $x^m = f(y^m)$ is to move along a specific world-line, does not uniquely define y^m. The existence of this coordinate freedom allows us to impose the harmonic gauge condition on the metric density $\tilde{g}^{mn} = \sqrt{-g}g^{mn}$ in the local frame:

$$\frac{\partial}{\partial y^m_a}\big(\sqrt{-g}g^{mn}\big) = 0. \tag{52}$$

The vanishing of the covariant derivative of the metric tensor (i.e., $(\sqrt{-g}g^{mn})_{;m} = (\partial/\partial y^m_a)(\sqrt{-g}g^{mn}) + \sqrt{-g}g^{kl}\Gamma^n_{kl} = 0$, where Γ^n_{mk} are the Christoffel symbols associated with the metric tensor g_{mn} in the moving frame $\{y^m_a\}$, and the semicolon represents covariant differentiation with respect to the coordinates, allows us to present Eq. (52) in the following equivalent form:

$$g^{kl}\Gamma^n_{kl} = 0. \tag{53}$$

Furthermore, remembering the rules of transformation of the Christoffel symbols under general coordinate transformations, we can see that Eq. (53) is equivalent to imposing the harmonic conditions on the transformation functions in Eqs. (37)–(38):

$$\Box_y x^m = 0, \tag{54}$$

where $\Box_y = (\sqrt{-g})^{-1}(\partial/\partial y^m_a)(\sqrt{-g}g^{mn}\partial/\partial y^n_a)$ denotes the d'Alembertian with respect to $\{y^n_a\}$ acting on x^m, which are treated as individual scalar functions.

Conversely, we also impose the harmonic gauge conditions on the inverse transformation:

$$\frac{\partial}{\partial x^m}\big(\sqrt{-g}g^{mn}\big) = 0 \qquad \text{and} \qquad \Box_x y^m_a = 0, \tag{55}$$

where $\Box_x = (\sqrt{-g})^{-1}(\partial/\partial x^m)(\sqrt{-g}g^{mn}\partial/\partial x^n)$ denotes the d'Alembertian with respect to global coordinates $\{x_m\}$ acting on the scalar functions y^m_a.

Therefore, on the one hand the harmonic gauge imposes restrictions on the partial derivatives of the metric tensor, as seen in Eq. (52). On the other hand, it restricts the choice of admissible coordinate transformations only to those that satisfy the harmonic equation (54). These two consequences of imposing the harmonic gauge will be used to establish the structure that the metric tensor, which will be given by Eqs. (126)–(128), must satisfy under the harmonic coordinate transformations (37)–(38). The harmonic gauge also constrains the form of the transformation functions $(\mathcal{K}_a, \mathcal{L}_a, \mathcal{Q}^\alpha_a)$. Below we explore this important dual role of the harmonic gauge in more detail.

To study the harmonic gauge conditions we will use a complete form of the metric tensor (7)–(9), where the component $g_{0\alpha}$ has the term of c^{-1} order, i.e., $g_{0\alpha} = c^{-1}g^{[1]}_{0\alpha} + c^{-3}g^{[3]}_{0\alpha} + \mathcal{O}(c^{-5})$, where $g^{[1]}_{0\alpha}$ has inertial nature [16]. As a result, the harmonic gauge conditions (52) yield the following set of partial differential equations for $\tilde{g}^{mn}$:

$$\begin{aligned}
\frac{\partial}{\partial y^0_a}\Big\{c^{-2}\tilde{g}^{[2]00} + \mathcal{O}(c^{-4})\Big\} + \frac{\partial}{\partial y^\epsilon_a}\Big\{c^{-1}\tilde{g}^{[1]\epsilon 0} + c^{-3}\tilde{g}^{[3]\epsilon 0} + \mathcal{O}(c^{-5})\Big\} &= 0, &(56)\\
\frac{\partial}{\partial y^0_a}\Big\{c^{-1}\tilde{g}^{[1]0\alpha} + \mathcal{O}(c^{-3})\Big\} + \frac{\partial}{\partial y^\epsilon_a}\Big\{c^{-2}\tilde{g}^{[2]\epsilon\alpha} + \mathcal{O}(c^{-4})\Big\} &= 0, &(57)
\end{aligned}$$

where we use bracketed superscript indices in the form $\tilde{g}^{[k]mn}$ to denote order terms with respect to inverse powers of c^{-k}, as discussed in the previous section. Grouping terms by order in these equations, we obtain the following set of equations on the metric components:

$$\begin{aligned}
\frac{\partial}{\partial y^\epsilon_a}\tilde{g}^{[1]\epsilon 0} &= \mathcal{O}(c^{-4}), &(58)\\
c\frac{\partial}{\partial y^0_a}\tilde{g}^{[2]00} + \frac{\partial}{\partial y^\epsilon_a}\tilde{g}^{[3]\epsilon 0} &= \mathcal{O}(c^{-2}), &(59)\\
c\frac{\partial}{\partial y^0_a}\tilde{g}^{[1]0\alpha} + \frac{\partial}{\partial y^\epsilon_a}\tilde{g}^{[2]\epsilon\alpha} &= \mathcal{O}(c^{-2}), &(60)
\end{aligned}$$

where the metric density components in the expressions above can be expressed via the covariant metric as

$$\tilde{g}^{[1]0\alpha} = -\gamma^{\alpha\nu}g^{[1]}_{0\nu}, \tag{61}$$

$$
\begin{aligned}
\tilde{g}^{[2]00} &= -\frac{1}{2}\Big\{g^{[2]}_{00} - \gamma^{\epsilon\lambda}\Big(g^{[2]}_{\epsilon\lambda} + g^{[1]}_{0\epsilon}g^{[1]}_{0\lambda}\Big)\Big\}, & (62)\\
\tilde{g}^{[3]0\alpha} &= -\gamma^{\alpha\nu}\Big\{g^{[3]}_{0\nu} + \frac{1}{2}g^{[1]}_{0\nu}\Big(\gamma^{\epsilon\lambda}\big(g^{[2]}_{\epsilon\lambda} + g^{[1]}_{0\epsilon}g^{[1]}_{0\lambda}\big) - g^{[2]}_{00}\Big) - \gamma^{\epsilon\lambda}g^{[1]}_{0\epsilon}g^{[2]}_{\nu\lambda}\Big\}, & (63)\\
\tilde{g}^{[2]\alpha\beta} &= -\gamma^{\alpha\epsilon}\gamma^{\beta\lambda}\Big(g^{[2]}_{\epsilon\lambda} - g^{[1]}_{0\epsilon}g^{[1]}_{0\lambda}\Big) + \frac{1}{2}\gamma^{\alpha\beta}\Big(g^{[2]}_{00} + \gamma^{\epsilon\lambda}\big(g^{[2]}_{\epsilon\lambda} - g^{[1]}_{0\epsilon}g^{[1]}_{0\lambda}\big)\Big). & (64)
\end{aligned}
$$

As a result, the harmonic conditions Eqs. (58)–(60) take the following form:

$$
\begin{aligned}
\frac{\partial}{\partial y_{a\epsilon}}g^{[1]}_{0\epsilon} &= \mathcal{O}(c^{-4}), & (65)\\
\frac{1}{2}c\frac{\partial}{\partial y^0_a}\Big\{g^{[2]}_{00} - \gamma^{\epsilon\lambda}\Big(g^{[2]}_{\epsilon\lambda} + g^{[1]}_{0\epsilon}g^{[1]}_{0\lambda}\Big)\Big\} + \frac{\partial}{\partial y_{a\nu}}\Big\{g^{[3]}_{0\nu} + \frac{1}{2}g^{[1]}_{0\nu}\Big(\gamma^{\epsilon\lambda}\big(g^{[2]}_{\epsilon\lambda} + g^{[1]}_{0\epsilon}g^{[1]}_{0\lambda}\big) - g^{[2]}_{00}\Big) - \gamma^{\epsilon\lambda}g^{[1]}_{0\epsilon}g^{[2]}_{\nu\lambda}\Big\} &= \mathcal{O}(c^{-2}), & (66)\\
c\frac{\partial}{\partial y^0_a}g^{[1]}_{0\alpha} + \frac{\partial}{\partial y_{a\beta}}\Big\{g^{[2]}_{\alpha\beta} - g^{[1]}_{0\alpha}g^{[1]}_{0\beta} - \frac{1}{2}\gamma_{\alpha\beta}\Big(g^{[2]}_{00} + \gamma^{\epsilon\lambda}\big(g^{[2]}_{\epsilon\lambda} - g^{[1]}_{0\epsilon}g^{[1]}_{0\lambda}\big)\Big)\Big\} &= \mathcal{O}(c^{-2}). & (67)
\end{aligned}
$$

A general solution to Eq. (65) may be presented in the form below:

$$
g^{[1]}_{0\alpha}(y_a) = \mu_\alpha(y^0_a) + \epsilon_{\alpha\nu}(y^0_a)y^\nu_a + \mathcal{O}(c^{-4}), \tag{68}
$$

where μ_α and $\epsilon_{\alpha\nu}$ (with $\epsilon_{\alpha\nu} = -\epsilon_{\nu\alpha}$) are arbitrary functions of the time-like coordinate y^0_a. The function μ_α may be interpreted as a rate of time shift in the chosen reference frame, while $\epsilon_{\alpha\nu}$ represents a rotation and shear of the coordinate axis. By an additional gauge transformation of the coordinates to a co-moving reference frame, we can always eliminate μ_α without loss of generality. We can also require the proper reference frame of a moving observer to exhibit no Newtonian rotation or shear of its coordinate axes. In addition to the fact that the leading term in the mixed components of the energy-momentum tensor is of the order $S_{0\alpha} \propto c$ (see Eq. (135) below), in terms of the metric tensor, these two physical conditions are equivalent to a requirement that all mixed components of the metric tensor of the proper reference frame at the c^{-1} order vanish. In other words, $g^{[1]}_{0\alpha}$ has the form:

$$
g^{[1]}_{0\alpha} = \mathcal{O}(c^{-4}). \tag{69}
$$

This condition leads to a significant simplification of the system of equations (65)–(67):

$$
\begin{aligned}
g^{[1]}_{0\alpha} &= \mathcal{O}(c^{-4}), & (70)\\
\frac{1}{2}c\frac{\partial}{\partial y^0_a}\Big\{g^{[2]}_{00} - \gamma^{\epsilon\lambda}g^{[2]}_{\epsilon\lambda}\Big\} + \frac{\partial}{\partial y_{a\nu}}g^{[3]}_{0\nu} &= \mathcal{O}(c^{-2}), & (71)\\
\frac{\partial}{\partial y_{a\beta}}\Big\{g^{[2]}_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}\Big(g^{[2]}_{00} + \gamma^{\epsilon\lambda}g^{[2]}_{\epsilon\lambda}\Big)\Big\} &= \mathcal{O}(c^{-2}). & (72)
\end{aligned}
$$

We note that, typically, researchers use only the last two of these equations. This is done to constrain the degrees of freedom in the Ricci tensor. For these purposes the harmonic gauge conditions reduce to the following two equations:

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial y^0_a}\Big(\gamma^{\epsilon\nu}g^{[2]}_{\epsilon\nu} - g^{[2]}_{00}\Big) - \frac{\partial}{\partial y_{a\nu}}g^{[3]}_{0\nu} &= \mathcal{O}(c^{-2}), & (73)\\
\frac{1}{2}\frac{\partial}{\partial y_{a\alpha}}\Big(g^{[2]}_{00} + \gamma^{\epsilon\nu}g^{[2]}_{\epsilon\nu}\Big) - \gamma^{\alpha\mu}\frac{\partial}{\partial y_{a\nu}}g^{[2]}_{\mu\nu} &= \mathcal{O}(c^{-2}). & (74)
\end{aligned}
$$

However, as we shall see shortly, the harmonic gauge conditions offer more than is typically used. Below we shall explore these possibilities. For this, by formally integrating Eq. (72), we get

$$
g^{[2]}_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}\Big(g^{[2]}_{00} + \gamma^{\epsilon\lambda}g^{[2]}_{\epsilon\lambda}\Big) = \mu_{\alpha\beta}(y^0) + \mathcal{O}(c^{-2}), \tag{75}
$$

where $\mu_{\alpha\beta}$ is an arbitrary function of the time-like coordinate y^0 only, representing a uniform scale expansion of the spatial part of the symmetric metric g_{mn}. After some algebra, Eq. (75) may be rewritten in the following form

$$
g^{[2]}_{\alpha\beta} + \gamma_{\alpha\beta}g^{[2]}_{00} = \mu_{\alpha\beta}(y^0_a) - \gamma_{\alpha\beta}\gamma^{\epsilon\lambda}\mu_{\epsilon\lambda}(y^0_a) + \mathcal{O}(c^{-2}). \tag{76}
$$

The asymptotic boundary conditions (36) require that the arbitrary function of time $\mu_{\alpha\beta}$ that enters Eq. (76) must be zero. Therefore, the boundary conditions require $\mu_{\alpha\beta}(y^0_a) = 0$, thus, the chosen coordinates are isotropic, we are led to the following form of the gauge conditions (70)–(72):

$$
\begin{aligned}
g^{[1]}_{0\alpha} &= \mathcal{O}(c^{-4}), & (77)\\
2c\frac{\partial}{\partial y^0_a}g^{[2]}_{00} + \frac{\partial}{\partial y_{a\nu}}g^{[3]}_{0\nu} &= \mathcal{O}(c^{-2}), & (78)\\
g^{[2]}_{\alpha\beta} + \gamma_{\alpha\beta}g^{[2]}_{00} &= \mathcal{O}(c^{-2}). & (79)
\end{aligned}
$$

Equivalently, for the contravariant form of the metric from Eq. (55), we obtain:

$$
\begin{aligned}
g^{[1]0\alpha} &= \mathcal{O}(c^{-4}), & (80)\\
2c\frac{\partial}{\partial x^0}g^{[2]00} + \frac{\partial}{\partial x^\nu}g^{[3]0\nu} &= \mathcal{O}(c^{-2}), & (81)\\
g^{[2]\alpha\beta} + \gamma^{\alpha\beta}g^{[2]00} &= \mathcal{O}(c^{-2}). & (82)
\end{aligned}
$$

These sets of gauge conditions form the foundation of our method of constructing a proper reference frame of a moving observer in the gravitational N-body problem that will be discussed next.

E. The form of the functions of the harmonic coordinate transformations

We now explore the alternative form of the harmonic gauge given by Eq. (54). Substituting the coordinate transformations (37)–(38) into Eq. (54), we can see that the harmonic gauge conditions restrict the coordinate transformation functions $\mathcal{K}_a, \mathcal{L}_a$ and $\mathcal{Q}^\alpha_a$ such that they must satisfy the following set of second order partial differential equations:

$$
\begin{aligned}
\gamma^{\epsilon\lambda}\frac{\partial^2\mathcal{K}_a}{\partial y^\epsilon_a\partial y^\lambda_a} &= \mathcal{O}(c^{-4}), & (83)\\
c^2\frac{\partial^2\mathcal{K}_a}{\partial {y^0_a}^2} + \gamma^{\epsilon\lambda}\frac{\partial^2\mathcal{L}_a}{\partial y^\epsilon_a\partial y^\lambda_a} &= \mathcal{O}(c^{-2}), & (84)\\
a^\alpha_{a_0} + \gamma^{\epsilon\lambda}\frac{\partial^2\mathcal{Q}^\alpha_a}{\partial y^\epsilon_a\partial y^\lambda_a} &= \mathcal{O}(c^{-2}). & (85)
\end{aligned}
$$

Analogously, for the inverse transformation functions we obtain:

$$
\begin{aligned}
\gamma^{\epsilon\lambda}\frac{\partial^2\hat{\mathcal{K}}_a}{\partial x^\epsilon\partial x^\lambda} &= \mathcal{O}(c^{-4}), & (86)\\
c^2\frac{\partial^2\hat{\mathcal{K}}_a}{\partial {x^0}^2} + \gamma^{\epsilon\lambda}\frac{\partial^2\hat{\mathcal{L}}_a}{\partial x^\epsilon\partial x^\lambda} &= \mathcal{O}(c^{-2}), & (87)\\
-\,a^\alpha_{a_0} + \gamma^{\epsilon\lambda}\frac{\partial^2\hat{\mathcal{Q}}^\alpha_a}{\partial x^\epsilon\partial x^\lambda} &= \mathcal{O}(c^{-2}). & (88)
\end{aligned}
$$

The general solution to these elliptic-type equations can be written in the form of a Taylor series expansion in terms of irreducible Cartesian tensors, which are symmetric trace-free (STF) tensors. Furthermore, the solutions for the functions $\mathcal{K}_a, \mathcal{L}_a$ and $\mathcal{Q}^\alpha_a$ in Eqs. (83)–(85) consist of two parts: a fundamental solution of the homogeneous Laplace equation and a particular solution of the inhomogeneous Poisson equation (except for Eq. (83), which is homogeneous). We discard the part of the fundamental solution that has a singularity at the origin of the local coordinates, where $y^\alpha = 0$, as non-physical.

As discussed at the beginning of this section, Eqs. (77)–(79) provide valuable constraints on the form of the metric tensor. As a matter of fact, these equations provide two additional conditions on $\mathcal{K}_a$ and $\mathcal{Q}^\alpha_a$. It follows from Eqs. (77), (79) and the form of the metric tensor g_{mn} given by Eqs. (126)–(128) (which are yet to be discussed in Sec. III A) that these two functions must also satisfy the following two first order partial differential equations:

$$
\frac{1}{c}\frac{\partial\mathcal{K}_a}{\partial y^\alpha_a} + v_{a_0\alpha} = \mathcal{O}(c^{-4}), \qquad (89)
$$

$$\frac{1}{c}\frac{\partial\mathcal{K}_a}{\partial y_a^\alpha}\frac{1}{c}\frac{\partial\mathcal{K}_a}{\partial y_a^\beta}+\gamma_{\alpha\lambda}\frac{\partial\mathcal{Q}_a^\lambda}{\partial y_a^\beta}+\gamma_{\beta\lambda}\frac{\partial\mathcal{Q}_a^\lambda}{\partial y_a^\alpha}+2\gamma_{\alpha\beta}\Big(\frac{\partial\mathcal{K}_a}{\partial y_a^0}+\tfrac{1}{2}v_{a_0\epsilon}v_{a_0}^\epsilon\Big) \;=\; \mathcal{O}(c^{-2}), \tag{90}$$

whereas the inverse functions $\hat{\mathcal{K}}_a$ and $\hat{\mathcal{Q}}_a^\alpha$ satisfy

$$\gamma^{\alpha\epsilon}\frac{1}{c}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^\epsilon}-v_{a_0}^\alpha \;=\; \mathcal{O}(c^{-4}), \tag{91}$$

$$v_{a_0}^\alpha v_{a_0}^\beta+\gamma^{\alpha\lambda}\frac{\partial\hat{\mathcal{Q}}_a^\beta}{\partial x^\lambda}+\gamma^{\beta\lambda}\frac{\partial\hat{\mathcal{Q}}_a^\alpha}{\partial x^\lambda}+2\gamma_{\alpha\beta}\Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}+\tfrac{1}{2}\gamma^{\epsilon\lambda}\frac{1}{c}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^\epsilon}\frac{1}{c}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^\lambda}\Big) \;=\; \mathcal{O}(c^{-2}). \tag{92}$$

The two sets of partial differential equations on $\mathcal{K}_a,\mathcal{L}_a$ and $\mathcal{Q}_a^\alpha$ given by Eqs. (83)–(85) and (89)–(90) can be used to determine the general structure of $\mathcal{K}_a$ and $\mathcal{Q}_a^\alpha$. Similarly, Eqs. (86)–(88) and (91)–(92) determine the general structure of $\hat{\mathcal{K}}_a$ and $\hat{\mathcal{Q}}_a^\alpha$. These will be required in order to solve the gravitational field equations. However, we must first develop the solution to the N-body problem in the quasi-inertial barycentric coordinate reference system.

F. Solution for the gravitational interaction term the global frame

In order to find the gravitational interaction term $h_{\rm int}^{mn}(x)$ in Eq. (51) and, thus, to determine the solution for the N-body problem in the barycentric frame, we will use the field equations (4) of the general theory of relativity. To do this, first of all we need to obtain the components of the metric tensor (51) in the barycentric reference system. This could be done by using the solution for the unperturbed one-body problem presented by Eq. (35), the transformation rules for coordinate base vectors given by Eqs.(42)–(43). Using all these expressions in Eq. (51) results in the following structure of the metric components in the global (barycentric) reference frame:

$$\begin{aligned} g^{00}(x) \;=\; & 1+\frac{2}{c^2}\sum_b\Big\{w_b(y_b(x))\big(1-\frac{2}{c^2}v_{b_0\epsilon}v_{b_0}^\epsilon\big)-\frac{4}{c^2}v_{b_0\epsilon}w_b^\epsilon(y_b(x))+ \\ & +\frac{2}{c^2}w_b(y_b(x))\Big(\frac{\partial\mathcal{K}_b}{\partial y_b^0}+\tfrac{1}{2}v_{b_0\epsilon}v_{b_0}^\epsilon\Big)+\frac{1}{c^2}w_b^2(y_b(x))\Big\}+\frac{1}{c^4}h_{\rm int}^{[4]00}(x)+\mathcal{O}(c^{-6}), \end{aligned} \tag{93}$$

$$g^{0\alpha}(x) \;=\; \frac{4}{c^3}\sum_b\Big\{w^\alpha(y_b(x))+v_{b_0}^\alpha w_b(y_b(x))\Big\}+\mathcal{O}(c^{-5}), \tag{94}$$

$$g^{\alpha\beta}(x) \;=\; \gamma^{\alpha\beta}-\gamma^{\alpha\beta}\frac{2}{c^2}\sum_b w_b(y_b(x))+\mathcal{O}(c^{-4}). \tag{95}$$

Next, we need to specify the "source term" for the N-body problem on the right-hand side of Eqs. (35). Similarly to (51), this term may be written in the following form:

$$S^{mn}(x)=\sum_{b=1}^N\frac{\partial x^m}{\partial y_b^k}\frac{\partial x^n}{\partial y_b^l}\;S_b^{kl}(y_b(x)), \tag{96}$$

where S_b^{kl} is the "source term" for a particular body b which was used to derive the one-body solution Eqs. (18)–(21).

By using the source term (96) and the transformation rules for coordinate base vectors given by Eqs. (42)–(43), and definitions Eqs. (18)–(21), we obtain the components of the source term S^{mn} in the barycentric reference frame:

$$S^{00}(x) \;=\; \frac{1}{2}c^2\sum_b\Big\{\Big(1-\frac{2}{c^2}v_{b_0\epsilon}v_{b_0}^\epsilon\Big)\sigma_b(y_b(x))-\frac{4}{c^2}v_{b_0\epsilon}\sigma_b^\epsilon(y_b(x))+\frac{2}{c^2}\big(\frac{\partial\mathcal{K}_b}{\partial y_b^0}+\tfrac{1}{2}v_{b_0\epsilon}v_{b_0}^\epsilon\big)\sigma_b(y_b(x))+\mathcal{O}(c^{-4})\Big\}, \tag{97}$$

$$S^{0\alpha}(x) \;=\; c\sum_b\Big\{\sigma_b^\alpha(y_b^k(x))+v_{b_0}^\alpha\sigma_b(y_b(x))+\mathcal{O}(c^{-2})\Big\}, \tag{98}$$

$$S^{\alpha\beta}(x) \;=\; -\gamma^{\alpha\beta}\frac{1}{2}c^2\sum_b\Big\{\sigma_b(y_b(x))+\mathcal{O}(c^{-2})\Big\}. \tag{99}$$

The components of the Ricci tensor in the global frame can be directly calculated from Eqs. (12)–(14):

$$R^{00}(x) \;=\; \frac{1}{2}\Box_x\Big(c^{-2}g^{[2]00}+c^{-4}\Big\{g^{[4]00}-\tfrac{1}{2}\big(g^{[2]00}\big)^2\Big\}\Big)+c^{-4}\frac{1}{2}\frac{\partial^2}{\partial x^\epsilon\partial x^\lambda}g^{[2]00}\Big(g^{[2]\epsilon\lambda}+\gamma^{\epsilon\lambda}g^{[2]00}\Big)+\mathcal{O}(c^{-6}), \tag{100}$$

$$
\begin{aligned}
R^{0\alpha}(x) &= c^{-3}\frac{1}{2}\Delta_x g^{[3]0\alpha} + \mathcal{O}(c^{-5}), &(101)\\
R^{\alpha\beta}(x) &= c^{-2}\frac{1}{2}\Delta_x g^{[2]\alpha\beta} + \mathcal{O}(c^{-4}), &(102)
\end{aligned}
$$

where $\Delta_x = \gamma^{\epsilon\lambda}\partial^2/\partial x^\epsilon \partial x^\lambda$ is the Laplace operator with respect to $\{x^k\}$.

Substituting the components (93)–(95) of the metric tensor and the components (97)–(99) of the source term into the field equations (4) of general relativity with the Ricci tensor (100)–(102), we obtain the gravitational field equations of the general theory of relativity in the barycentric reference frame:

$$
\begin{aligned}
\Box_x\Big\{\sum_b\Big[\big(1-\frac{2}{c^2}v_{b_0\epsilon}v^\epsilon_{b_0}\big)w_b(y_b(x)) - \frac{4}{c^2}v_{b_0\epsilon}w^\epsilon_b(y_b(x)) \;&+\; \frac{2}{c^2}w_b(y_b(x))\Big(\frac{\partial\mathcal{K}_b}{\partial y^0_b}+\tfrac{1}{2}v_{b_0\epsilon}v^\epsilon_{b_0}\Big)+\frac{1}{c^2}w^2_b(y_b(x))\Big] - \\
-\frac{1}{c^2}\Big(\sum_b w_b(y_b(x))\Big)^2+\frac{1}{2c^2}h^{[4]00}_{\rm int}(x)\Big\} &= 4\pi G\sum_b\Big\{\Big(1-\frac{2}{c^2}v_{b_0\epsilon}v^\epsilon_{b_0}\Big)\sigma_b(y_b(x)) - \frac{4}{c^2}v_{b_0\epsilon}\sigma^\epsilon_b(y_b(x)) + \\
&\quad + \frac{2}{c^2}\big(\frac{\partial\mathcal{K}_b}{\partial y^0_b}+\tfrac{1}{2}v_{b_0\epsilon}v^\epsilon_{b_0}\big)\sigma_b(y_b(x))\Big\}+\mathcal{O}(c^{-4}), \qquad (103)\\
\Delta_x\sum_b\Big\{w^\alpha_b(y_b(x))+v^\alpha_{b_0}w_b(y_b(x))\Big\} &= 4\pi G\sum_b\Big\{\sigma^\alpha_b(y_b(x))+v^\alpha_{b_0}\sigma_b(y_b(x))\Big\}+\mathcal{O}(c^{-2}), \qquad (104)\\
\Delta_x\sum_b w_b(y_b(x)) &= 4\pi G\sum_b \sigma_b(y_b(x))+\mathcal{O}(c^{-2}). \qquad (105)
\end{aligned}
$$

We observe that Eqs. (104) and (105) are satisfied identically. We will focus on Eq. (103) with the aim to determine the interaction term. For these purposes, we use Eqs. (104) and (105) and rewrite (103) in the following form:

$$
\begin{aligned}
\Box_x\Big\{\sum_b\Big[w_b(y_b(x)) \;+\; \frac{2}{c^2}\Big(\frac{\partial\mathcal{K}_b}{\partial y^0_b}+\tfrac{1}{2}v_{b_0\epsilon}v^\epsilon_{b_0}\Big)w_b(y_b(x))+\frac{1}{c^2}w^2_b(y_b(x))\Big]-\frac{1}{c^2}\Big(\sum_b w_b(y_b(x))\Big)^2+\frac{1}{2c^2}h^{[4]00}_{\rm int}(x)\Big\} &= \\
= 4\pi G\sum_b\Big\{\sigma_b(y_b(x))+\frac{2}{c^2}\Big(\frac{\partial\mathcal{K}_b}{\partial y^0_b}+\tfrac{1}{2}v_{b_0\epsilon}v^\epsilon_{b_0}\Big)\sigma_b(y_b(x))\Big\}+\mathcal{O}(c^{-4}). \qquad (106)
\end{aligned}
$$

To analyze this equation, it is helpful to express the d'Alembert operator $\Box_x$ via its counterpart $\Box_{y_a}$ expressed in coordinates y_a associated with body a. Using Eqs. (37)–(38), one easily obtains the following relation:

$$
\begin{aligned}
\frac{\partial^2}{\partial {x^0}^2}+\gamma^{\epsilon\lambda}\frac{\partial^2}{\partial x^\epsilon\partial x^\lambda} = \frac{\partial^2}{\partial {y^0_a}^2}+\gamma^{\epsilon\lambda}\frac{\partial^2}{\partial y^\epsilon_a\partial y^\lambda_a} \;&-\; \frac{1}{c^2}\Big\{a^\mu_{a_0}+\gamma^{\epsilon\lambda}\frac{\partial^2\mathcal{Q}^\mu_a}{\partial y^\epsilon_a\partial y^\lambda_a}\Big\}\frac{\partial}{\partial y^\mu_a} - \\
&-\; \frac{1}{c^2}\Big\{v^\epsilon_{a_0}v^\lambda_{a_0}+\gamma^{\epsilon\mu}\frac{\partial\mathcal{Q}^\lambda_a}{\partial y^\mu_a}+\gamma^{\lambda\mu}\frac{\partial\mathcal{Q}^\epsilon_a}{\partial y^\mu_a}\Big\}\frac{\partial^2}{\partial y^\epsilon_a\partial y^\lambda_a}+\mathcal{O}(c^{-4}). \qquad (107)
\end{aligned}
$$

The terms of c^{-2} order on the right-hand side of this expression can be further simplified with the help of harmonic gauge conditions. Indeed, using Eqs. (85) and (90) the expression above takes the following form:

$$
\frac{\partial^2}{\partial {x^0}^2}+\gamma^{\epsilon\lambda}\frac{\partial^2}{\partial x^\epsilon\partial x^\lambda} \;=\; \frac{\partial^2}{\partial {y^0_a}^2}+\gamma^{\epsilon\lambda}\frac{\partial^2}{\partial y^\epsilon_a\partial y^\lambda_a}+\frac{2}{c^2}\Big(\frac{\partial\mathcal{K}_a}{\partial y^0_a}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)\gamma^{\epsilon\lambda}\frac{\partial^2}{\partial y^\epsilon_a\partial y^\lambda_a}+\mathcal{O}(c^{-4}). \qquad (108)
$$

The last equation may be given in the equivalent form:

$$
\Box_x \;=\; \Big\{1+\frac{2}{c^2}\Big(\frac{\partial\mathcal{K}_a}{\partial y^0_a}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)+\mathcal{O}(c^{-4})\Big\}\Box_{y_a}. \qquad (109)
$$

Therefore, taking into account that, for any given y_b, the following relation holds

$$
\Box_{y_b}w_b(y_b(x)) \;=\; 4\pi G\sigma_b(y_b(x))+\mathcal{O}(c^{-4}), \qquad (110)
$$

therefore, it follows from Eqs. (109) and (110), the following result is valid for any body b:

$$
\Box_x w_b(y_b(x)) \;=\; 4\pi G\sigma_b(y_b(x))\Big\{1+\frac{2}{c^2}\Big(\frac{\partial\mathcal{K}_b}{\partial y^0_b}+\tfrac{1}{2}v_{b_0\epsilon}v^\epsilon_{b_0}\Big)+\mathcal{O}(c^{-4})\Big\}. \qquad (111)
$$

With the help of Eq. (111) we can now rewrite Eq. (106) as below

$$\Box_x\Big\{\sum_b\Big[\frac{2}{c^2}\Big(\frac{\partial\mathcal{K}_b}{\partial y^0_b}+\tfrac{1}{2}v_{b_0\epsilon}v^\epsilon_{b_0}\Big)w_b(y_b(x))+\frac{1}{c^2}w^2_b(y_b(x))\Big]-\frac{1}{c^2}\Big(\sum_b w_b(y_b(x))\Big)^2+\frac{1}{2c^2}h^{[4]00}_{\rm int}(x)\Big\}=\mathcal{O}(c^{-4}). \quad (112)$$

This equation can be solved for $h^{[4]00}_{\rm int}(x)$. The solution to this partial differential equation is composed of two families: one is a particular solution that can be found by equating the entire expression under the D'Alembert operator to zero, $h^{[4]00}_{\rm int(0)}(x)$, while the other is a family of solutions involving STF tensors that are divergent at spatial infinity:

$$h^{[4]00}_{\rm int}(x)=h^{[4]00}_{\rm int(0)}(x)+\sum_{k\geq0}\frac{1}{k!}\delta h^{00}_{{\rm int}\,\mu_1\ldots\mu_k}(x^0)x^{\mu_1}\cdots x^{\mu_k}+\mathcal{O}(|x^\mu|^K)+\mathcal{O}(c^{-2}), \quad (113)$$

where $h^{00}_{{\rm int}\,\mu_1\ldots\mu_k}(x^0)$ being arbitrary STF tensors that depend only on the time-like coordinate x^0. Requiring that the solution to this equation will be asymptotically flat and and that there is no gravitational radiation coming from outside the system Eq. (36) namely

$$\lim_{\substack{r\to\infty\\ t+r/c={\rm const}}} h^{[4]00}_{\rm int}(x)=0 \quad {\rm and} \quad \lim_{\substack{r\to\infty\\ t+r/c={\rm const}}}\big[(rh^{[4]00}_{\rm int})_{,r}+(rh^{[4]00}_{\rm int})_{,0}\big]=0, \quad (114)$$

implies that $h^{00}_{{\rm int}\,\mu_1\ldots\mu_k}(x^0)=0$, for all k. Therefore, Eq. (113) yields the following solution for the interaction term:

$$h^{[4]00}_{\rm int}(x)=2\Big\{\Big(\sum_b w_b(y_b(x))\Big)^2-\sum_b w^2_b(y_b(x))\Big\}-4\sum_b w_b(y_b(x))\Big(\frac{\partial\mathcal{K}_b}{\partial y^0_b}+\tfrac{1}{2}v_{b_0\epsilon}v^\epsilon_{b_0}\Big)+\mathcal{O}(c^{-2}). \quad (115)$$

The first two terms on the right-hand side of the equation above represent the usual composition of the gravitational interaction term for N-body system. The last term is the coupling of the local inertia field to the local gravity potential transformed to the global coordinates. The presence of this coupling reduces the total energy in the system, acting as a binding energy for the combined system of gravity and inertia.

Substituting the interaction term Eq. (115) into the $g^{00}(x)$ component (93) of the metric tensor we obtain the following expression for the metric tensor in the global frame:

$$g^{00}(x) = 1+\frac{2}{c^2}\sum_b w_b(x)+\frac{2}{c^4}\Big(\sum_b w_b(x)\Big)^2+\mathcal{O}(c^{-6}), \quad (116)$$

$$g^{0\alpha}(x) = \frac{4}{c^3}\sum_b w^\alpha_b(x)+\mathcal{O}(c^{-5}), \quad (117)$$

$$g^{\alpha\beta}(x) = \gamma^{\alpha\beta}-\gamma^{\alpha\beta}\frac{2}{c^2}\sum_b w_b(x)+\mathcal{O}(c^{-4}), \quad (118)$$

where the gravitational scalar, $w_b(x)$, and vector, $w^\alpha_b(x)$, potentials of a body b transformed to the global coordinates $\{x^m\}$ have the following form:

$$w_b(x) = \big(1-\frac{2}{c^2}v_{b_0\epsilon}v^\epsilon_{b_0}\big)w_b(y_b(x))-\frac{4}{c^2}v_{b_0\epsilon}w^\epsilon_b(y_b(x))+\mathcal{O}(c^{-4}), \quad (119)$$

$$w^\alpha_b(x) = w^\alpha_b(y_b(x))+v^\alpha_{b_0}w_b(y_b(x))+\mathcal{O}(c^{-2}). \quad (120)$$

It is convenient to express these potentials in terms of the integrals over the volumes of the bodies in the global coordinates. To do that we use Eqs. (111) and (104) to determine the following solutions for the scalar $w_b(y_b(x))$ and vector $w^\alpha_b(y_b(x))$ potentials that satisfy boundary conditions (36):

$$\begin{aligned} w_b(y_b(x)) &= G\int d^3x'\frac{\sigma_b(y_b(x'))}{|\mathbf{x}-\mathbf{x}'|}+\frac{1}{2c^2}G\frac{c^2\partial^2}{\partial x^{0\,2}}\int d^3x'\sigma_b(y_b(x'))|\mathbf{x}-\mathbf{x}'|+\\ &\quad +\frac{2G}{c^2}\int d^3x'\frac{\sigma_b(y_b(x'))}{|\mathbf{x}-\mathbf{x}'|}\Big(\frac{\partial\mathcal{K}'_b}{\partial y^0_b}+\tfrac{1}{2}v_{b_0\epsilon}v^\epsilon_{b_0}\Big)+\mathcal{O}(c^{-3}), \end{aligned} \quad (121)$$

$$w^\alpha_b(y_b(x)) = G\int d^3x'\frac{\sigma^\alpha_b(y_b(x'))}{|\mathbf{x}-\mathbf{x}'|}+\mathcal{O}(c^{-2}). \quad (122)$$

Substituting these results in Eqs. (119)–(120) in the form of the integrals over the body's volume leads to the following expression for gravitational potentials of the body b in the global coordinates of inertial reference frame:

$$
\begin{aligned}
w_b(x) &= \big(1-\frac{2}{c^2}v_{b_0\epsilon}v^\epsilon_{b_0}\big)G\int d^3x'\frac{\sigma_b(y_b(x'))}{|\mathbf{x}-\mathbf{x}'|}-\frac{4}{c^2}v_{b_0\epsilon}G\int d^3x'\frac{\sigma^\epsilon_b(y_b(x'))}{|\mathbf{x}-\mathbf{x}'|}+\\
&\quad+\frac{1}{2c^2}G\frac{c^2\partial^2}{\partial x^{0\,2}}\int d^3x'\sigma_b(y_b(x'))|\mathbf{x}-\mathbf{x}'|+\frac{2G}{c^2}\int d^3x'\frac{\sigma_b(y_b(x'))}{|\mathbf{x}-\mathbf{x}'|}\Big(\frac{\partial\mathcal{K}'_b}{\partial y^0_b}+\tfrac{1}{2}v_{b_0\epsilon}v^\epsilon_{b_0}\Big)+\mathcal{O}(c^{-3}), \qquad (123)\\
w^\alpha_b(x) &= G\int d^3x'\frac{\sigma^\alpha_b(y_b(x'))}{|\mathbf{x}-\mathbf{x}'|}+v^\alpha_{b_0}G\int d^3x'\frac{\sigma_b(y_b(x'))}{|\mathbf{x}-\mathbf{x}'|}+\mathcal{O}(c^{-2}). \qquad (124)
\end{aligned}
$$

We have obtained a solution for the gravitational field equations for the N-body problem in the post-Newtonian approximation. In the global (barycentric) reference frame this solution is represented by the metric tensor given by Eqs. (116)–(118) together with the scalar and vector potentials given by Eqs. (123) and (124). Note that the form of the metric (116)–(118) is identical to that of the isolated one-body problem (35), except that these two structurally-identical metric tensors have different potentials. Such form-invariance of the metric tensor is due to harmonic gauge conditions and the covariant structure of the total metric sought in the form of Eq. (51) together with the covariantly-superimposed form of the N-body source term Eq. (96).

In the following section we transform this solution to the local frame, verify that this solution satisfies the equations of general relativity in that frame, and improve the form of this solution using the harmonic gauge.

III. TRANSFORMATION TO THE LOCAL FRAME

We begin this section by investigating the transformation properties of the covariant form of the metric tensor from the global frame to the local frame.

A. Gravitational field equations in the local frame

The metric tensor in the local frame can be obtained by applying the standard rules of tensor transformations to the metric tensor in the global frame given by Eqs. (116)–(118):

$$
g_{mn}(y_a)=\frac{\partial x^k}{\partial y^m_a}\frac{\partial x^l}{\partial y^n_a}\,g_{kl}(x(y_a)). \qquad (125)
$$

Using the coordinate transformations given by Eqs. (37)–(38) together with Eqs. (42)–(43), we determine the form of the metric tensor in the local reference frame associated with a body a:

$$
\begin{aligned}
g_{00}(y_a) &= 1+\frac{2}{c^2}\Big\{\frac{\partial\mathcal{K}_a}{\partial y^0_a}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big\}+\frac{2}{c^4}\Big\{\frac{\partial\mathcal{L}_a}{\partial y^0_a}+\tfrac{1}{2}\Big(\frac{\partial\mathcal{K}_a}{\partial y^0_a}\Big)^2+cv_{a_0\epsilon}\frac{\partial\mathcal{Q}^\epsilon_a}{\partial y^0_a}-\Big(\frac{\partial\mathcal{K}_a}{\partial y^0_a}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)^2\Big\}-\\
&\quad-\frac{2}{c^2}\sum_b\Big\{\Big(1-\frac{2}{c^2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)w_b(x(y_a))+\frac{4}{c^2}v_{a_0\epsilon}w^\epsilon_b(x(y_a))\Big\}+\\
&\quad+\frac{2}{c^4}\Big(\sum_b w_b(x(y_a))-\frac{\partial\mathcal{K}_a}{\partial y^0_a}-\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)^2+\mathcal{O}(c^{-6}), \qquad (126)\\
g_{0\alpha}(y_a) &= \frac{1}{c}\Big(\frac{1}{c}\frac{\partial\mathcal{K}_a}{\partial y^\alpha_a}+v_{a_0\alpha}\Big)-\frac{4}{c^3}\gamma_{\alpha\epsilon}\sum_b\Big(w^\epsilon_b(x(y_a))-v^\epsilon_{a_0}w_b(x(y_a))\Big)+\\
&\quad+\frac{1}{c^3}\Big\{\frac{1}{c}\frac{\partial\mathcal{L}_a}{\partial y^\alpha_a}+c\gamma_{\alpha\epsilon}\frac{\partial\mathcal{Q}^\epsilon_a}{\partial y^0_a}+v_{a_0\lambda}\frac{\partial\mathcal{Q}^\lambda_a}{\partial y^\alpha_a}+\frac{1}{c}\frac{\partial\mathcal{K}_a}{\partial y^\alpha_a}\frac{\partial\mathcal{K}_a}{\partial y^0_a}\Big\}+\mathcal{O}(c^{-5}), \qquad (127)\\
g_{\alpha\beta}(y_a) &= \gamma_{\alpha\beta}+\frac{1}{c^2}\Big\{\frac{1}{c}\frac{\partial\mathcal{K}_a}{\partial y^\alpha_a}\frac{1}{c}\frac{\partial\mathcal{K}_a}{\partial y^\beta_a}+\gamma_{\alpha\lambda}\frac{\partial Q^\lambda_a}{\partial y^\beta_a}+\gamma_{\beta\lambda}\frac{\partial Q^\lambda_a}{\partial y^\alpha_a}\Big\}+\gamma_{\alpha\beta}\frac{2}{c^2}\sum_b w_b(x(y_a))+\mathcal{O}(c^{-4}). \qquad (128)
\end{aligned}
$$

The form of the metric in the local reference frame given by Eqs. (126)–(128) still has a significant number of degrees of freedom reflecting the presence of the yet arbitrary transformation functions. Some of these degrees of freedom can be eliminated by imposing the harmonic gauge conditions. Eqs. (77) and (79) can provide valuable constraints on the form of the metric tensor in the local frame. Indeed, using the actual structure of the metric expressed by

Eqs. (126)–(128), the gauge conditions (77)–(79) and Eqs. (89)–(90), we can now use these two sets of constraints on the transformation functions and to improve the form of the metric tensor in the local frame. As a result, the metric tensor in the harmonic coordinates of the local frame has the following form:

$$\begin{aligned}
g_{00}(y_a) &= 1+\frac{2}{c^2}\Big\{\frac{\partial\mathcal{K}_a}{\partial y^0_a}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big\}+\frac{2}{c^4}\Big\{\frac{\partial\mathcal{L}_a}{\partial y^0_a}+\tfrac{1}{2}\Big(\frac{\partial\mathcal{K}_a}{\partial y^0_a}\Big)^2+cv_{a_0\epsilon}\frac{\partial\mathcal{Q}^\epsilon_a}{\partial y^0_a}-\Big(\frac{\partial\mathcal{K}_a}{\partial y^0_a}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)^2\Big\}- \\
&-\frac{2}{c^2}\sum_b\Big\{\Big(1-\frac{2}{c^2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)w_b(x(y_a))+\frac{4}{c^2}v_{a_0\epsilon}w^\epsilon_b(x(y_a))\Big\}+ \\
&+\frac{2}{c^4}\Big(\sum_b w_b(x(y_a))-\frac{\partial\mathcal{K}_a}{\partial y^0_a}-\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)^2+\mathcal{O}(c^{-6}), \qquad (129)\\
g_{0\alpha}(y_a) &= -\frac{4}{c^3}\gamma_{\alpha\epsilon}\sum_b\Big(w^\epsilon_b(x(y_a))-v^\epsilon_{a_0}w_b(x(y_a))\Big)+ \\
&+\frac{1}{c^3}\Big\{\frac{1}{c}\frac{\partial\mathcal{L}_a}{\partial y^\alpha_a}+c\gamma_{\alpha\epsilon}\frac{\partial\mathcal{Q}^\epsilon_a}{\partial y^0_a}+v_{a_0\lambda}\frac{\partial\mathcal{Q}^\lambda_a}{\partial y^\alpha_a}-v_{a_0\alpha}\frac{\partial\mathcal{K}_a}{\partial y^0_a}\Big\}+\mathcal{O}(c^{-5}), \qquad (130)\\
g_{\alpha\beta}(y_a) &= \gamma_{\alpha\beta}+\gamma_{\alpha\beta}\frac{2}{c^2}\Big\{\sum_b w_b(x(y_a))-\frac{\partial\mathcal{K}_a}{\partial y^0_a}-\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big\}+\mathcal{O}(c^{-4}). \qquad (131)
\end{aligned}$$

The contravariant components of the source term in the local reference frame, $S_{mn}(y_a)$, are related to that in the global frame, $S_{mn}(x)$, via the usual tensor transformation:

$$S_{mn}(y_a)=\frac{\partial x^k}{\partial y^m_a}\frac{\partial x^l}{\partial y^n_a}\;S_{kl}(x(y_a)). \qquad (132)$$

On the other hand, given the definition for the scalar and vector densities (which are given by Eqs. (19)–(21)), for which we need covariant components of the source term S^{mn}, we can determine $S_{mn}(y_a)$ in the local frame via the covariant components S^{mn} in the following form

$$S_{mn}(y_a)=g_{mk}(y_a)g_{nl}(y_a)S^{kl}(y_a)=g_{mk}(y_a)g_{nl}(y_a)\frac{\partial y^k_a}{\partial x^p}\frac{\partial y^l_a}{\partial x^q}\;S^{pq}(x(y_a)), \qquad (133)$$

where $\partial y^k_a/\partial x^p$ are given by Eqs. (49)–(50), which lead to the following expression for the components of the source term $S_{mn}(y_a)$:

$$\begin{aligned}
S_{00}(y_a) &= \Big(1+\frac{2}{c^2}g^{[2]}_{00}(y_a)+\mathcal{O}(c^{-4})\Big)\times \\
&\times\frac{1}{2}c^2\sum_b\Big\{\Big(1-\frac{2}{c^2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)\sigma_b(x(y_a))+\frac{4}{c^2}v_{a_0\epsilon}\sigma^\epsilon_b(x(y_a))+\frac{2}{c^2}\Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)\sigma_b(x(y_a))+\mathcal{O}(c^{-4})\Big\}, \qquad (134)\\
S_{0\alpha}(y_a) &= \gamma_{\alpha\lambda}c\sum_b\Big\{\sigma^\lambda_b(x(y_a))-v^\lambda_{a_0}\sigma_b(x(y_a))+\mathcal{O}(c^{-2})\Big\}, \qquad (135)\\
S_{\alpha\beta}(y_a) &= -\gamma_{\alpha\beta}\frac{1}{2}c^2\Big\{\sum_b\sigma_b(x(y_a))+\mathcal{O}(c^{-2})\Big\}, \qquad (136)
\end{aligned}$$

where $\hat{\mathcal{K}}_a$ comes as a part of the rules Eqs. (49)–(50) that are developed for the inverse transformations $y^m=f^m(x^k)$.

Similarly, to compute the Ricci tensor in the global frame, we use the following expression:

$$R_{mn}(y_a)=g_{mk}(y_a)g_{nl}(y_a)R^{kl}(y_a), \qquad (137)$$

with $R^{kl}(y_a)$ being expressed in terms of the covariant metric g_{mn}, $R^{kl}[g_{mn}(y_a)]$, namely:

$$\begin{aligned}
R_{00}(y_a) &= \Big(1+\frac{2}{c^2}g^{[2]}_{00}(x)+\mathcal{O}(c^{-2})\Big) \\
&\times\Big(-\frac{1}{2}\Box_{y_a}\Big(c^{-2}g^{[2]}_{00}+c^{-4}\Big\{g^{[4]}_{00}-\tfrac{1}{2}\big(g^{[2]}_{00}\big)^2\Big\}\Big)+c^{-4}\frac{1}{2}\frac{\partial^2}{\partial y_{a\epsilon}\partial y_{a\lambda}}g^{[2]}_{00}\Big(g^{[2]}_{\epsilon\lambda}+\gamma_{\epsilon\lambda}g^{[2]}_{00}\Big)+\mathcal{O}(c^{-6})\Big), \qquad (138)\\
R_{0\alpha}(y_a) &= -c^{-3}\frac{1}{2}\Delta_{y_a}g^{[3]}_{0\alpha}+\mathcal{O}(c^{-5}), \qquad (139)
\end{aligned}$$

$$R_{\alpha\beta}(y_a) \ = \ -c^{-2}\frac{1}{2}\Delta_{y_a} g^{[2]}_{\alpha\beta} + \mathcal{O}(c^{-4}), \tag{140}$$

where $\Delta_{y_a} = \gamma^{\epsilon\lambda}\partial^2/\partial y^\epsilon_a \partial y^\lambda_a$ is the Laplace operator with respect to the coordinates $\{y^k_a\}$.

The metric tensor given by Eqs. (129)–(131) and the condition given by Eq. (79) allow us to present the gravitational field equations (4) of the general theory of relativity in local coordinates in the following form:

$$\begin{aligned}
&\Box_{y_a}\Big[\sum_b \Big\{\Big(1-\frac{2}{c^2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)w_b(x(y_a)) \ + \ \frac{4}{c^2}v_{a_0\epsilon}w^\epsilon_b(x(y_a))\Big\} - \\
&\qquad -\frac{\partial\mathcal{K}_a}{\partial y^0_a} - \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0} - \frac{2}{c^2}\Big\{\frac{\partial\mathcal{L}_a}{\partial y^0_a} \ + \ \tfrac{1}{2}\Big(\frac{\partial\mathcal{K}_a}{\partial y^0_a}\Big)^2 + cv_{a_0\epsilon}\frac{\partial\mathcal{Q}^\epsilon_a}{\partial y^0_a} - \Big(\frac{\partial\mathcal{K}_a}{\partial y^0_a} + \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)^2\Big\} + \mathcal{O}(c^{-2})\Big] = \\
&= 4\pi G\sum_b \Big\{\Big(1-\frac{2}{c^2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)\sigma_b(x(y_a)) \ + \ \frac{4}{c^2}v_{a_0\epsilon}\sigma^\epsilon_b(x(y_a)) + \frac{2}{c^2}\big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0} + \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\big)\sigma_b(x(y_a)) + \mathcal{O}(c^{-4})\Big\},
\end{aligned} \tag{141}$$

$$\begin{aligned}
&\Delta_{y_a}\Big[\gamma_{\alpha\epsilon}\sum_b\Big(w^\epsilon_b(x(y_a)) - v^\epsilon_{a_0}w_b(x(y_a))\Big) \ - \ \tfrac{1}{4}\Big\{\frac{1}{c}\frac{\partial\mathcal{L}_a}{\partial y^\alpha_a} + c\gamma_{\alpha\epsilon}\frac{\partial\mathcal{Q}^\epsilon_a}{\partial y^0_a} + v_{a_0\lambda}\frac{\partial\mathcal{Q}^\lambda_a}{\partial y^\alpha_a} - v_{a_0\alpha}\frac{\partial\mathcal{K}_a}{\partial y^0_a}\Big\} + \mathcal{O}(c^{-5})\Big] = \\
&\qquad = \ 4\pi G\gamma_{\alpha\lambda}\sum_b\Big\{\sigma^\lambda_b(x(y_a)) - v^\lambda_{a_0}\sigma_b(x(y_a))\Big\} + \mathcal{O}(c^{-2}),
\end{aligned} \tag{142}$$

$$\Delta_{y_a}\Big[\sum_b w_b(x(y_a)) - \frac{\partial\mathcal{K}_a}{\partial y^0_a} \ - \ \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0} + \mathcal{O}(c^{-2})\Big] = 8\pi G\gamma_{\alpha\beta}\sum_b\sigma_b(x(y_a)) + \mathcal{O}(c^{-2}). \tag{143}$$

Using the fact that $\Delta_{y_a} = \Delta_x + \mathcal{O}(c^{-2})$ and the gauge equations (83)–(85) we can show that Eqs. (142) and (143) are identically satisfied. Furthermore, Eq. (141) reduces to

$$\Box_{y_a}\sum_b w_b(x(y_a)) = 4\pi G\sum_b\Big\{\sigma_b(x(y_a)) + \frac{2}{c^2}\big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0} + \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\big)\sigma_b(x(y_a)) + \mathcal{O}(c^{-4})\Big\}. \tag{144}$$

We can verify the validity of this equation in a manner similar to the derivation shown by Eqs. (107) and (108): We can express the d'Alembertian in the local coordinates $\Box_{y_a}$ via global coordinates $\{x^m\}$ as below:

$$\begin{aligned}
\frac{\partial^2}{\partial {y^0_a}^2} + \gamma^{\epsilon\lambda}\frac{\partial^2}{\partial y^\epsilon_a\partial y^\lambda_a} = \frac{\partial^2}{\partial {x^0}^2} + \gamma^{\epsilon\lambda}\frac{\partial^2}{\partial x^\epsilon\partial x^\lambda} \ + \ \frac{1}{c^2}\Big\{a^\mu_{a_0} - \gamma^{\epsilon\lambda}\frac{\partial^2\hat{\mathcal{Q}}^\mu_a}{\partial x^\epsilon\partial x^\lambda}\Big\}\frac{\partial}{\partial x^\mu} - \\
- \ \frac{1}{c^2}\Big\{v^\epsilon_{a_0}v^\lambda_{a_0} + \gamma^{\epsilon\mu}\frac{\partial\hat{\mathcal{Q}}^\lambda_a}{\partial x^\mu} + \gamma^{\lambda\mu}\frac{\partial\hat{\mathcal{Q}}^\epsilon_a}{\partial x^\mu}\Big\}\frac{\partial^2}{\partial x^\epsilon\partial x^\lambda} + \mathcal{O}(c^{-4}),
\end{aligned} \tag{145}$$

where $\partial y^k_a/\partial x^p$ needed to derive Eq. (145) are given by Eqs. (49)–(50). Terms of order c^{-2} that are present on the right-hand side of this equation can be simplified using the harmonic gauge conditions (88) and (92), so that Eq. (145) takes the form:

$$\frac{\partial^2}{\partial {y^0_a}^2} + \gamma^{\epsilon\lambda}\frac{\partial^2}{\partial y^\epsilon_a\partial y^\lambda_a} \ = \ \frac{\partial^2}{\partial {x^0}^2} + \gamma^{\epsilon\lambda}\frac{\partial^2}{\partial x^\epsilon\partial x^\lambda} + \frac{2}{c^2}\Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0} + \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)\gamma^{\epsilon\lambda}\frac{\partial^2}{\partial x^\epsilon\partial x^\lambda} + \mathcal{O}(c^{-4}). \tag{146}$$

The last equation may be given in the equivalent form:

$$\Box_{y_a} \ = \ \Big\{1 + \frac{2}{c^2}\Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0} + \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big) + \mathcal{O}(c^{-4})\Big\}\Box_x. \tag{147}$$

As a side result, comparing this expression with Eq. (109), we see that the following approximate relation holds:

$$\Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0} + \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big) = -\Big(\frac{\partial\mathcal{K}_a}{\partial y^0_a} + \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big) + \mathcal{O}(c^{-2}). \tag{148}$$

Using Eq. (31) we see that the following relation satisfied for any $\{y_a\}$:

$$\Box_x w_b(x(y_a)) \ = \ 4\pi G\sigma_b(x(y_a)) + \mathcal{O}(c^{-4}), \tag{149}$$

and relying on (147), we observe that the following relation also holds:

$$\Box_{y_a} w_b(x(y_a)) \;=\; 4\pi G\sigma_b(x(y_a))\Big\{1+\frac{2}{c^2}\Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)+\mathcal{O}(c^{-4})\Big\}, \tag{150}$$

which proves Eq. (144). As a result, with the help of Eqs. (83)–(85), and also (150), we verified that Eqs. (141)–(143) are satisfied identically.

In the next subsection, we use the harmonic gauge conditions to constrain this remaining freedom which would lead to a particular structure of both the metric tensor and the coordinate transformations.

B. The form of the metric tensor in the local frame

Eqs. (129)–(131) represent the metric tensor η_{mn} given by Eqs. (77)–(79) in the coordinates of a local reference frame that satisfies the harmonic gauge. This set of gauge conditions forms the foundation of our method of constructing a proper reference frame of a gravitationally accelerated observer. As a result, the metric representing spacetime in the proper frame may be presented in the following elegant isotropic form that depends only on two harmonic potentials[2]:

$$g_{00}(y_a) \;=\; 1-\frac{2}{c^2}w(y_a)+\frac{2}{c^4}w^2(y_a)+O(c^{-6}), \tag{151}$$

$$g_{0\alpha}(y_a) \;=\; -\gamma_{\alpha\lambda}\frac{4}{c^3}w^\lambda(y_a)+O(c^{-5}), \tag{152}$$

$$g_{\alpha\beta}(y_a) \;=\; \gamma_{\alpha\beta}+\gamma_{\alpha\beta}\frac{2}{c^2}w(y_a)+O(c^{-4}), \tag{153}$$

where the total (gravitation plus inertia) local scalar $w(y^k_a)$ and vector $w^\alpha(y^k_a)$ potentials have the following form:

$$w(y_a) \;=\; \sum_b w_b(y_a)-\frac{\partial\mathcal{K}_a}{\partial y^0_a}-\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}-\frac{1}{c^2}\Big\{\frac{\partial\mathcal{L}_a}{\partial y^0_a}+cv_{a_0\epsilon}\frac{\partial\mathcal{Q}^\epsilon_a}{\partial y^0_a}+\tfrac{1}{2}\Big(\frac{\partial\mathcal{K}_a}{\partial y^0_a}\Big)^2-\Big(\frac{\partial\mathcal{K}_a}{\partial y^0_a}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)^2\Big\}+O(c^{-4}), \tag{154}$$

$$w^\alpha(y_a) \;=\; \sum_b w^\alpha_b(y_a)-\tfrac{1}{4}\Big\{\gamma^{\alpha\epsilon}\frac{1}{c}\frac{\partial\mathcal{L}_a}{\partial y^\epsilon_a}+c\frac{\partial\mathcal{Q}^\alpha_a}{\partial y^0_a}+\gamma^{\alpha\epsilon}v_{a_0\lambda}\frac{\partial\mathcal{Q}^\lambda_a}{\partial y^\epsilon_a}-v^\alpha_{a_0}\frac{\partial\mathcal{K}_a}{\partial y^0_a}\Big\}+O(c^{-2}), \tag{155}$$

with gravitational potentials $w_b(y_a)$ and $w^\alpha_b(y_a)$ transform between the global and local frames as below

$$w_b(y_a) \;=\; \Big(1-\frac{2}{c^2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)w_b(x(y_a))+\frac{4}{c^2}v_{a_0\epsilon}w^\epsilon_b(x(y_a))+\mathcal{O}(c^{-4}), \tag{156}$$

$$w^\alpha_b(y_a) \;=\; w^\alpha_b(x(y_a))-v^\alpha_{a_0}w_b(x(y_a))+\mathcal{O}(c^{-2}). \tag{157}$$

It follows from Eq. (78) that potentials $w_b(y_a)$ and $w^\alpha_b(y_a)$ satisfy the continuity equation:

$$c\frac{\partial w}{\partial y^0_a}+\frac{\partial w^\epsilon}{\partial y^\epsilon_a}=\mathcal{O}(c^{-2}). \tag{158}$$

With the help of Eqs. (83)–(85) and (89)–(90), we can verify that the new potentials w and w^α satisfy the following post-Newtonian harmonic equations:

$$\Box_{y_a}w(y_a)=4\pi G\sum_b\Big\{\Big(1-\frac{2}{c^2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)\sigma_b(x(y_a))+\frac{4}{c^2}v_{a_0\epsilon}\sigma^\epsilon_b(x(y_a))+\frac{2}{c^2}\Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)\sigma_b(x(y_a))+\mathcal{O}(c^{-4})\Big\}, \tag{159}$$

$$\Delta_{y_a}w^\alpha(y_a)=4\pi G\sum_b\Big\{\sigma^\alpha_b(x(y_a))-v^\alpha_{a_0}\sigma_b(x(y_a))+\mathcal{O}(c^{-2})\Big\}. \tag{160}$$

---

[2] Alternatively, one can write the metric tensor, given by Eqs. (151)–(153), in the following equivalent harmonic form:

$$g_{00}(y_a) \;=\; \frac{1-c^{-2}w(y_a)}{1+c^{-2}w(y_a)}+O(c^{-6}),\qquad g_{0\alpha}(y_a)=-\gamma_{\alpha\lambda}\frac{4}{c^3}w^\lambda(y_a)+O(c^{-5}),\qquad g_{\alpha\beta}(y_a)=\gamma_{\alpha\beta}\Big(1+c^{-2}w(y_a)\Big)^2+O(c^{-4}).$$

One may verify that according to Eqs. (150) and (142), the scalar and vector potentials can be presented as functions of the local coordinates only:

$$
\begin{aligned}
w_b(x(y_a)) &= G\int d^3y'_a\frac{\sigma_b(x(y'_a))}{|\mathbf{y}_a-\mathbf{y}'_a|}+\frac{1}{2c^2}G\frac{c^2\partial^2}{\partial y_a^{0\,2}}\int d^3y'_a\sigma_b(x(y'_a))|\mathbf{y}_a-\mathbf{y}'_a|+\\
&\qquad +\frac{2G}{c^2}\int d^3y'_a\frac{\sigma_b(x(y'_a))}{|\mathbf{y}_a-\mathbf{y}'_a|}\Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}+\tfrac{1}{2}v_{a_0\epsilon}v^{\epsilon}_{a_0}\Big)+\mathcal{O}(c^{-4}), && (161)\\
w^{\alpha}_b(x(y_a)) &= G\int d^3y'_a\frac{\sigma^{\alpha}_b(x(y'_a))}{|\mathbf{y}_a-\mathbf{y}'_a|}+\mathcal{O}(c^{-2}). && (162)
\end{aligned}
$$

These expressions allow us to present the scalar and vector potentials, given by Eqs. (156)–(157), via barycentric densities given as functions of the local coordinates:

$$
\begin{aligned}
w_b(y_a) &= \big(1-\frac{2}{c^2}v_{a_0\epsilon}v^{\epsilon}_{a_0}\big)G\int d^3y'_a\frac{\sigma_b(x(y'_a))}{|\mathbf{y}_a-\mathbf{y}'_a|}+\frac{4}{c^2}v_{a_0\epsilon}G\int d^3y'_a\frac{\sigma^{\epsilon}_b(x(y'_a))}{|\mathbf{y}_a-\mathbf{y}'_a|}+\\
&\quad+\frac{1}{2c^2}G\frac{c^2\partial^2}{\partial y_a^{0\,2}}\int d^3y'_a\sigma_b(x(y'_a))|\mathbf{y}_a-\mathbf{y}'_a|+\frac{2G}{c^2}\int d^3y'_a\frac{\sigma_b(x(y'_a))}{|\mathbf{y}_a-\mathbf{y}'_a|}\Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}+\tfrac{1}{2}v_{a_0\epsilon}v^{\epsilon}_{a_0}\Big)+\mathcal{O}(c^{-3}), && (163)\\
w^{\alpha}_b(y_a) &= G\int d^3y'_a\frac{\sigma^{\alpha}_b(x(y'_a))}{|\mathbf{y}_a-\mathbf{y}'_a|}-v^{\alpha}_{a_0}G\int d^3y'_a\frac{\sigma_b(x(y'_a))}{|\mathbf{y}_a-\mathbf{y}'_a|}+\mathcal{O}(c^{-2}). && (164)
\end{aligned}
$$

One can also present the transformation rules for the relativistic gravitational potentials from the proper coordinates $\{y_b\}$ associated with body b to the coordinates $\{y_a\}$ associated with body a. With the help of Eqs. (119) and (120), we can express the potentials $w_b(x)$ and $w^{\alpha}_b(x)$ as functions of $\{x(y_a)\}$ and substitute the results into Eqs. (156) and (157), as shown below:

$$
\begin{aligned}
w_b(y_a) &= \Big(1-\frac{2}{c^2}v_{ba_0\epsilon}v^{\epsilon}_{ba_0}\Big)w_b(y_b(y_a))+\frac{4}{c^2}v_{ba_0\epsilon}w^{\epsilon}_b(y_b(y_a))+\mathcal{O}(c^{-4}), && (165)\\
w^{\alpha}_b(y_a) &= w^{\alpha}_b(y_b(y_a))-v^{\alpha}_{ba_0}w_b(y_b(y_a))+\mathcal{O}(c^{-2}), && (166)
\end{aligned}
$$

with $v^{\alpha}_{ba_0}=v^{\alpha}_{a_0}-v^{\alpha}_{b_0}$ being the relative velocity between the two bodies. Assuming the existence of the coordinate transformations between the two bodies in the form

$$
\begin{aligned}
y^0_b &= y^0_a+c^{-2}\mathcal{K}_{ba}(y^0_a,y^{\epsilon}_a)+\mathcal{O}(c^{-4}), && (167)\\
y^{\alpha}_b &= y^{\alpha}_a+z^{\alpha}_{ba_0}(y^0_a)+\mathcal{O}(c^{-2}), && (168)
\end{aligned}
$$

with function $\mathcal{K}_{ba}$ given [8] as $\mathcal{K}_{ba}(y^0_a,y^{\epsilon}_a)=\mathcal{K}_a(y^0_a,y^{\epsilon}_a)-\mathcal{K}_b(y^0_a,y^{\epsilon}_a+z^{\epsilon}_{ba_0})+\mathcal{O}(c^{-2})$, we note that the scalar and vector potentials, $w_b(y_b(y_a))$ and $w^{\alpha}_b(y_b(y_a))$ correspondingly, satisfy the following equations:

$$
\begin{aligned}
\Box_{y_a}w_b(y_b(y_a)) &= 4\pi G\sigma_b(y_b(y_a))\Big\{1-\frac{2}{c^2}\Big(\frac{\partial\mathcal{K}_{ba}}{\partial y^0_a}+\tfrac{1}{2}v_{ba_0\epsilon}v^{\epsilon}_{ba_0}\Big)+\mathcal{O}(c^{-4})\Big\}, && (169)\\
\Delta_{y_a}w^{\alpha}_b(y_b(y_a)) &= 4\pi G\sigma^{\alpha}_b(y_b(y_a))+\mathcal{O}(c^{-2}). && (170)
\end{aligned}
$$

In Eq. (169), similarly to Eq. (148) we used the following identity:

$$
\Big(\frac{\partial\mathcal{K}_{ab}}{\partial y^0_b}+\tfrac{1}{2}v_{ba_0\epsilon}v^{\epsilon}_{ba_0}\Big)=-\Big(\frac{\partial\mathcal{K}_{ba}}{\partial y^0_a}+\tfrac{1}{2}v_{ba_0\epsilon}v^{\epsilon}_{ba_0}\Big)+\mathcal{O}(c^{-2}). \tag{171}
$$

Eqs. (169) and (170) have the following solutions

$$
\begin{aligned}
w_b(y_b(y_a)) &= G\int d^3y'_a\frac{\sigma_b(y_b(y'_a))}{|\mathbf{y}_a-\mathbf{y}'_a|}+\frac{1}{2c^2}G\frac{c^2\partial^2}{\partial y_a^{0\,2}}\int d^3y'_a\sigma_b(y_b(y'_a))|\mathbf{y}_a-\mathbf{y}'_a|-\\
&\qquad-\frac{2G}{c^2}\int d^3y'_a\frac{\sigma_b(y_b(y'_a))}{|\mathbf{y}_a-\mathbf{y}'_a|}\Big(\frac{\partial\mathcal{K}_{ba}}{\partial y^0_a}+\tfrac{1}{2}v_{ba_0\epsilon}v^{\epsilon}_{ba_0}\Big)+\mathcal{O}(c^{-4}), && (172)\\
w^{\alpha}_b(y_b(y_a)) &= G\int d^3y'_a\frac{\sigma^{\alpha}_b(y_b(y'_a))}{|\mathbf{y}_a-\mathbf{y}'_a|}+\mathcal{O}(c^{-2}). && (173)
\end{aligned}
$$

Therefore, finally, we obtain the following expressions for the scalar and vector gravitational potentials, expressed as integrals over the compact volumes of the bodies given in the local coordinates:

$$\begin{aligned} w_b(y_a) &= \big(1-\frac{2}{c^2}v_{ba_0\epsilon}v^\epsilon_{ba_0}\big)G\int d^3y'_a\frac{\sigma_b(y_b(y'_a))}{|\mathbf{y}_a-\mathbf{y}'_a|}+\frac{4}{c^2}v_{ba_0\epsilon}G\int d^3y'_a\frac{\sigma^\epsilon_b(y_b(y'_a))}{|\mathbf{y}_a-\mathbf{y}'_a|}+\\ &+\frac{1}{2c^2}G\frac{c^2\partial^2}{\partial y_a^{0\,2}}\int d^3y'_a\sigma_b(y_b(y'_a))|\mathbf{y}_a-\mathbf{y}'_a|-\frac{2G}{c^2}\int d^3y'_a\frac{\sigma_b(y_b(y'_a))}{|\mathbf{y}_a-\mathbf{y}'_a|}\Big(\frac{\partial\mathcal{K}_{ba}}{\partial y^0_a}+\tfrac{1}{2}v_{ba_0\epsilon}v^\epsilon_{ba_0}\Big)+\mathcal{O}(c^{-3}), \quad (174)\\ w^\alpha_b(y_a) &= G\int d^3y'_a\frac{\sigma^\alpha_b(y_b(y'_a))}{|\mathbf{y}_a-\mathbf{y}'_a|}-v^\alpha_{ba_0}G\int d^3y'_a\frac{\sigma_b(y_b(y'_a))}{|\mathbf{y}_a-\mathbf{y}'_a|}+\mathcal{O}(c^{-2}). \quad (175)\end{aligned}$$

We use the phrase *harmonic metric tensor* to describe the metric tensor (151)–(153), defined in terms of the harmonic potentials w and w^α, which in turn are given by Eqs. (154)–(155) and satisfy the harmonic equations (159)–(160).

C. Applying the harmonic gauge conditions to reconstruct the transformation functions

The second role of the harmonic conditions in our derivation is to put further constraints on coordinate transformations, taking us closer to determine in full the form of the functions $\mathcal{K}_a$, $\mathcal{Q}^\alpha_a$ and $\mathcal{L}_a$.

1. *Determining the structure of* $\mathcal{K}_a$

The general solution to Eq. (83) with regular behavior on the world-line (i.e., omitting terms divergent when $|\mathbf{y}_a|\to 0$ or solutions not differentiable at $|\mathbf{y}_a|=0$) can be given in the following form:

$$\mathcal{K}_a(y_a)=\kappa_{a_0}+\kappa_{a_0\mu}y^\mu_a+\delta\kappa_a(y_a)+\mathcal{O}(c^{-4}),\qquad \text{where}\qquad \delta\kappa_a(y_a)=\sum_{k=2}\frac{1}{k!}\kappa_{a_0\mu_1\ldots\mu_k}(y^0_a)y^{\mu_1}_a\cdots y^{\mu_k}_a+\mathcal{O}(c^{-4}), \quad (176)$$

with $\kappa_{a_0\mu_1\ldots\mu_k}(y^0_a)$ being STF tensors [24], which depend only on the time-like coordinate y^0_a. Substituting this form of the function $\mathcal{K}_a$ into Eq. (89), we the find solutions for $\kappa_{a_0\mu}$ and $\kappa_{a\mu_1\ldots\mu_k}$:

$$\kappa_{a_0\mu}=-cv_{a_0\mu}+\mathcal{O}(c^{-4}),\qquad \kappa_{a_0\mu_1\ldots\mu_k}=\mathcal{O}(c^{-4}),\qquad k\geq 2. \quad (177)$$

As a result, the function $\mathcal{K}_a$ that satisfies the harmonic gauge conditions is determined to be

$$\mathcal{K}_a(y_a)=\kappa_{a_0}-c(v_{a_0\mu}y^\mu_a)+\mathcal{O}(c^{-4}). \quad (178)$$

This expression completely fixes the spatial dependence of the function $\mathcal{K}_a$, but still has an arbitrary dependence on the time-like coordinate via the function $\kappa_{a_0}(y^0_a)$.

2. *Determining the structure of* $\mathcal{Q}^\alpha_a$

The general solution for the function $\mathcal{Q}^\alpha$ that satisfies Eq. (85) may be presented as a sum of a solution of the inhomogeneous Poisson equation and a solution of the homogeneous Laplace equation. In particular, solutions with regular behavior in the vicinity of the world-line may be given in the following form:

$$\mathcal{Q}^\alpha_a(y_a)=q^\alpha_{a_0}+q^\alpha_{a_0\mu}y^\mu_a+\tfrac{1}{2}q^\alpha_{a_0\mu\nu}y^\mu_ay^\nu_a+\delta\xi^\alpha_a(y_a)+\mathcal{O}(c^{-2}), \quad (179)$$

where $q^\alpha_{a_0\mu\nu}$ can be determined directly from Eq. (85) and the function $\delta\xi^\alpha_a$ satisfies Laplace's equation

$$\gamma^{\epsilon\lambda}\frac{\partial^2}{\partial y^\epsilon_a\partial y^\lambda_a}\delta\xi^\alpha_a \;=\; \mathcal{O}(c^{-2}). \quad (180)$$

We can see that Eq. (85) can be used to determine $q^\alpha_{a_0\mu\nu}$, but would leave the other terms in the equation unspecified. To determine these terms, we use Eq. (90) together with Eq. (89), and get:

$$v_{a_0\alpha}v_{a_0\beta}+\gamma_{\alpha\lambda}\frac{\partial\mathcal{Q}^\lambda_a}{\partial y^\beta_a}+\gamma_{\beta\lambda}\frac{\partial\mathcal{Q}^\lambda_a}{\partial y^\alpha_a}+2\gamma_{\alpha\beta}\Big(\frac{\partial\mathcal{K}_a}{\partial y^0_a}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big) \;=\; \mathcal{O}(c^{-2}). \quad (181)$$

Using the intermediate solution (178) for the function $\mathcal{K}_a$ in Eq. (181), we obtain the following equation for $\mathcal{Q}^\alpha_a$:

$$v_{a_0\alpha}v_{a_0\beta}+\gamma_{\alpha\lambda}\frac{\partial\mathcal{Q}^\lambda_a}{\partial y^\beta_a}+\gamma_{\beta\lambda}\frac{\partial\mathcal{Q}^\lambda_a}{\partial y^\alpha_a}+2\gamma_{\alpha\beta}\Big(\frac{\partial\kappa_{a_0}}{\partial y^0_a}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}-a_{a_0\epsilon}y^\epsilon_a\Big) \;=\; \mathcal{O}(c^{-2}). \tag{182}$$

A trial solution to Eq. (182) may be given in the following general from:

$$\begin{aligned}\mathcal{Q}^\alpha_a \;=\;& q^\alpha_{a_0}+c_1v^\alpha_{a_0}v_{a_0\epsilon}y^\epsilon_a+c_2v_{a_0\epsilon}v^\epsilon_{a_0}y^\alpha_a+c_3a^\alpha_{a_0}y_{a\epsilon}y^\epsilon_a+c_4a_{a_0\epsilon}y^\epsilon_ay^\alpha_a+c_5\big(\frac{\partial\kappa_{a_0}}{\partial y^0_a}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\big)y^\alpha_a+\\ &+\,y_{a\epsilon}\omega^{\epsilon\alpha}_{a_0}+\delta\xi^\alpha_a(y_a)+\mathcal{O}(c^{-2}),\end{aligned} \tag{183}$$

where $q^\alpha_{a_0}$ and the antisymmetric matrix $\omega^{\alpha\epsilon}_{a_0}=-\omega^{\epsilon\alpha}_{a_0}$ are functions of the time-like coordinate y^0_a; $c_1,...,c_5$ are constant coefficients; and $\delta\xi^\mu_a(y_a)$, given by Eq. (180), is at least of the third order in spatial coordinates y^μ_a, namely $\delta\xi^\mu_a(y_a)\propto\mathcal{O}(|y^\mu_a|^3)$. Direct substitution of Eq. (183) into Eq. (182) results in the following unique solution for the constant coefficients:

$$c_1=-\frac{1}{2},\quad c_2=0,\quad c_3=-\frac{1}{2},\quad c_4=1,\quad c_5=-1. \tag{184}$$

As a result, the function $\mathcal{Q}^\alpha_a$ has the following structure:

$$\mathcal{Q}^\alpha_a(y_a) \;=\; q^\alpha_{a_0}-\Big(\tfrac{1}{2}v^\alpha_{a_0}v^\epsilon_{a_0}+\omega^{\alpha\epsilon}_{a_0}+\gamma^{\alpha\epsilon}\big(\frac{\partial\kappa_{a_0}}{\partial y^0_a}+\tfrac{1}{2}v_{a_0\lambda}v^\lambda_{a_0}\big)\Big)y_{a\epsilon}+a_{a_0\epsilon}\Big(y^\alpha_ay^\epsilon_a-\tfrac{1}{2}\gamma^{\alpha\epsilon}y_{a\lambda}y^\lambda_a\Big)+\delta\xi^\alpha_a(y_a)+\mathcal{O}(c^{-2}), \tag{185}$$

where $q^\alpha_{a_0}$ and $\omega^{\alpha\epsilon}_{a_0}$ are yet to be determined.

By substituting Eq. (185) into Eq. (182), we see that the function $\delta\xi^\alpha_a(y_a)$ in Eq. (185) must satisfy the equation:

$$\frac{\partial}{\partial y_{a\alpha}}\delta\xi^\beta_a+\frac{\partial}{\partial y_{a\beta}}\delta\xi^\alpha_a=\mathcal{O}(c^{-2}). \tag{186}$$

We keep in mind that the function $\delta\xi^\alpha_a(y_a)$ must also satisfy Eq. (180). The solution to the partial differential equation (180) with regular behavior on the world-line (i.e., when $|\mathbf{y}_a|\to0$) can be given in powers of y^μ_a as

$$\delta\xi^\alpha_a(y_a)=\sum_{k\geq3}\frac{1}{k!}\delta\xi^\alpha_{a_0\,\mu_1...\mu_k}(y^0_a)y^{\mu_1}_a\cdots y^{\mu_k}_a+\mathcal{O}(|y^\mu_a|^K)+\mathcal{O}(c^{-2}), \tag{187}$$

where $\delta\xi^\alpha_{a_0\,\mu_1...\mu_k}(y^0_a)$ are STF tensors that depend only on time-like coordinate. Using the solution (187) in Eq. (186), we can see that $\delta\xi^\alpha_{0\,\mu_1...\mu_k}$ is also antisymmetric with respect to the index α and any of the spatial indices $\mu_1...\mu_k$. Combination of these two conditions suggests that $\delta\xi^\alpha_{0\,\mu_1...\mu_k}=0$ for all $k\geq3$, thus

$$\delta\xi^\alpha_a(y_a)=\mathcal{O}(c^{-2}). \tag{188}$$

Therefore, application of the harmonic gauge conditions leads to the following structure of the function $\mathcal{Q}^\alpha_a$:

$$\mathcal{Q}^\alpha_a(y_a) \;=\; q^\alpha_{a_0}-\Big(\tfrac{1}{2}v^\alpha_{a_0}v^\epsilon_{a_0}+\omega^{\alpha\epsilon}_{a_0}+\gamma^{\alpha\epsilon}\big(\frac{\partial\kappa_{a_0}}{\partial y^0_a}+\tfrac{1}{2}v_{a_0\lambda}v^\lambda_{a_0}\big)\Big)y_{a\epsilon}+a_{a_0\epsilon}\Big(y^\alpha_ay^\epsilon_a-\tfrac{1}{2}\gamma^{\alpha\epsilon}y_{a\lambda}y^\lambda_a\Big)+\mathcal{O}(c^{-2}), \tag{189}$$

where $q^\alpha_{a_0},\omega^{\alpha\epsilon}_{a_0}$ and κ_{a_0} are yet to be determined.

3. *Determining the structure of* $\mathcal{L}_a$

We now turn our attention to the second gauge condition on the temporal coordinate transformation, Eq. (84). Using the intermediate solution (178) of the function $\mathcal{K}_a$, we obtain the following equation for $\mathcal{L}_a$:

$$\gamma^{\epsilon\lambda}\frac{\partial^2\mathcal{L}_a}{\partial y^\epsilon_a\partial y^\lambda_a} \;=\; -c^2\frac{\partial^2\mathcal{K}_a}{\partial {y^0_a}^2}+\mathcal{O}(c^{-2})=c\big(v_{a_0\epsilon}a^\epsilon_{a_0}+\dot a_{a_0\epsilon}y^\epsilon_a\big)-c^2\frac{\partial}{\partial y^0_a}\Big(\frac{\partial\kappa_{a_0}}{\partial y^0_a}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)+\mathcal{O}(c^{-2}). \tag{190}$$

The general solution of Eq. (190) for $\mathcal{L}_a$ may be presented as a sum of a solution $\delta\mathcal{L}_a$ for the inhomogeneous Poisson equation and a solution $\delta\mathcal{L}_{a0}$ of the homogeneous Laplace equation. A trial solution of the inhomogeneous equation to this equation, $\delta\mathcal{L}_a$, is sought in the following form:

$$\delta\mathcal{L}_a=ck_1(v_{a_0\epsilon}a^\epsilon_{a_0})(y_{a\mu}y^\mu_a)+ck_2(\dot a_{a_0\epsilon}y^\epsilon_a)(y_{a\nu}y^\nu_a)-k_3c^2\frac{\partial}{\partial y^0_a}\Big(\frac{\partial\kappa_{a_0}}{\partial y^0_a}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)(y_{a\nu}y^\nu_a)+\mathcal{O}(c^{-2}), \tag{191}$$

where k_1, k_2, k_3 are some constants. Direct substitution of Eq. (191) into Eq. (190) results in the following values for these constants:

$$k_1 = \frac{1}{6}, \quad k_2 = \frac{1}{10}, \quad k_3 = \frac{1}{6}. \tag{192}$$

As a result, the solution for $\delta\mathcal{L}_a$ that satisfies the harmonic gauge conditions has the following form:

$$\delta\mathcal{L}_a(y_a) \ = \ \tfrac{1}{6}c\big(v_{a_0\epsilon}a^{\epsilon}_{a_0}\big)(y_{a\nu}y^{\nu}_a) + \tfrac{1}{10}c(\dot{a}_{a_0\epsilon}y^{\epsilon}_a)(y_{a\nu}y^{\nu}_a) - \tfrac{1}{6}c^2\frac{\partial}{\partial y^0_a}\Big(\frac{\partial\kappa_{a_0}}{\partial y^0_a} + \tfrac{1}{2}v_{a_0\epsilon}v^{\epsilon}_{a_0}\Big)(y_{a\nu}y^{\nu}_a) + \mathcal{O}(c^{-2}). \tag{193}$$

The solution for the homogeneous equation with regular behavior on the world-line (i.e., when $|y^{\mu}_a| \to 0$) may be presented as follows:

$$\mathcal{L}_{a0}(y_a) \ = \ \ell_{a_0}(y^0) + \ell_{a_0\lambda}(y^0_a)\, y^{\lambda}_a + \tfrac{1}{2}\ell_{a_0\lambda\mu}(y^0_a)\, y^{\lambda}_a y^{\mu}_a + \delta\ell_a(y_a) + \mathcal{O}(c^{-2}), \tag{194}$$

where $\ell_{a_0\lambda\mu}$ is an STF tensor of second rank and $\delta\ell_a$ is a function formed from similar STF tensors of higher order:

$$\delta\ell_a(y_a) = \sum_{k\geq 3}\frac{1}{k!}\delta\ell_{a_0\,\mu_1\ldots\mu_k}(y^0_a)y^{\mu_1}_a\cdots y^{\mu_k}_a + \mathcal{O}(|y^{\mu}_a|^K) + \mathcal{O}(c^{-2}). \tag{195}$$

Finally, the general solution of the Eq. (190) may be presented as a sum of the special solution $\delta\mathcal{L}_a$ of the inhomogeneous equation and the solution $\mathcal{L}_{a0}$ of the homogeneous equation $\Delta\mathcal{L}_a = 0$. Therefore, the general solution for the harmonic gauge equations for the function $\mathcal{L}_a(y_a) = \mathcal{L}_{a0} + \delta\mathcal{L}_a$ has the following form:

$$\begin{aligned}\mathcal{L}_a(y_a) \ &= \ \ell_{a_0} + \ell_{a_0\lambda}\, y^{\lambda}_a + \tfrac{1}{2}\ell_{a_0\lambda\mu}\, y^{\lambda}_a y^{\mu}_a + \delta\ell_a(y_a) + \\ &+ \ \tfrac{1}{6}c\big(v_{a_0\epsilon}a^{\epsilon}_{a_0}\big)(y_{a\nu}y^{\nu}_a) + \tfrac{1}{10}c(\dot{a}_{a_0\epsilon}y^{\epsilon}_a)(y_{a\nu}y^{\nu}_a) - \tfrac{1}{6}c^2\frac{\partial}{\partial y^0_a}\Big(\frac{\partial\kappa_{a_0}}{\partial y^0_a} + \tfrac{1}{2}v_{a_0\epsilon}v^{\epsilon}_{a_0}\Big)(y_{a\nu}y^{\nu}_a) + \mathcal{O}(c^{-2}).\end{aligned} \tag{196}$$

D. Harmonic functions of the coordinate transformation to a local frame

As a result of imposing the harmonic gauge conditions, we were able to determine the structure of the transformation functions $\mathcal{K}_a, \mathcal{Q}^{\alpha}_a$ and $\mathcal{L}_a$ that satisfy the harmonic gauge:

$$\mathcal{K}_a(y_a) \ = \ \kappa_{a_0} - c(v_{a_0\mu}y^{\mu}_a) + \mathcal{O}(c^{-4}), \tag{197}$$

$$\mathcal{Q}^{\alpha}_a(y_a) \ = \ q^{\alpha}_{a_0} - \Big(\tfrac{1}{2}v^{\alpha}_{a_0}v^{\epsilon}_{a_0} + \omega^{\alpha\epsilon}_{a_0} + \gamma^{\alpha\epsilon}\big(\frac{\partial\kappa_{a_0}}{\partial y^0_a} + \tfrac{1}{2}v_{a_0\lambda}v^{\lambda}_{a_0}\big)\Big)y_{a\epsilon} + a_{a_0\epsilon}\Big(y^{\alpha}_a y^{\epsilon}_a - \tfrac{1}{2}\gamma^{\alpha\epsilon}y_{a\lambda}y^{\lambda}_a\Big) + \mathcal{O}(c^{-2}), \tag{198}$$

$$\begin{aligned}\mathcal{L}_a(y_a) \ &= \ \ell_{a_0} + \ell_{a_0\lambda}\, y^{\lambda}_a + \tfrac{1}{2}\ell_{a_0\lambda\mu}\, y^{\lambda}_a y^{\mu}_a + \delta\ell_a(y_a) + \\ &+ \ \tfrac{1}{6}c\Big(\big(v_{a_0\epsilon}a^{\epsilon}_{a_0}\big) - c\frac{\partial}{\partial y^0_a}\big(\frac{\partial\kappa_{a_0}}{\partial y^0_a} + \tfrac{1}{2}v_{a_0\epsilon}v^{\epsilon}_{a_0}\big)\Big)(y_{a\nu}y^{\nu}_a) + \tfrac{1}{10}c(\dot{a}_{a_0\epsilon}y^{\epsilon}_a)(y_{a\nu}y^{\nu}_a) + \mathcal{O}(c^{-2}).\end{aligned} \tag{199}$$

Note that the harmonic gauge conditions allow us to reconstruct the structure of these functions with respect to spatial coordinate y^{μ}_a. However, the time-dependent functions $\kappa_{a_0}, q^{\alpha}_{a_0}, \omega^{\alpha\epsilon}_{a_0}$, $\ell_{a_0}, \ell_{a_0\lambda}, \ell_{a_0\lambda\mu}$, and $\delta\ell_{a_0\,\mu_1\ldots\mu_k}$ cannot be determined from the gauge conditions alone. We need to apply another set of conditions that would dynamically define the proper reference frame of a moving observer, thereby fixing these time-dependent functions. This procedure will be discussed in the following section.

IV. DYNAMICAL CONDITIONS FOR A PROPER REFERENCE FRAME

As we saw in the preceding section, introduction of the harmonic gauge allows one to determine the harmonic structure of the transformation functions $\mathcal{K}_a$, $\mathcal{L}_a$ and $\mathcal{Q}^{\alpha}_a$, but it was insufficient to completely determine these functions. To proceed further, we investigate the dynamics of a co-moving observer in the accelerated reference frame.

An observer that remains at rest with respect to the accelerating frame does so because of the balance between an external force and the fictitious frame-reaction force. If the external force is universal (in the sense of the weak equivalence principle) and in the case of complete balance of this fictitious frame-reaction force, there will be no net force acting on the observer allowing it to be at rest in what we shall call its proper reference frame. The motion of

the observer in this frame, then, will resemble a free-fall that follows a geodesic with respect to the metric g_{mn}. It would, thus, be natural to require that in order for the observer to be at rest in its proper reference frame associated with the local metric, the observer's ordinary relativistic linear three-momentum should be conserved. We can explore these conditions by writing down the Lagrangian of a test particle that represents the observer and find that it is possible to eliminate all the remaining unknown components of $\mathcal{K}_a$, $\mathcal{L}_a$ and $\mathcal{Q}^\alpha_a$.

A. 'Good' proper reference frame and local metric conditions

The metric tensor g_{mn}, given by Eqs. (151)–(153) allows one to study the dynamics of the reference frame that moves in response to the presence of the external force. The Lagrangian L of a test particle that moves in this system can be obtained directly from the metric g_{mn} and written as below [16, 20]:

$$\begin{aligned} L &= -mc^2\frac{ds}{dy^0_a} = -mc^2\Big(g_{mn}\frac{dy^m_a}{dy^0_a}\frac{dy^n_a}{dy^0_a}\Big)^{1/2} = \\ &= -mc^2\Big\{1+c^{-2}\Big(\tfrac{1}{2}v_\epsilon v^\epsilon - w\Big) + c^{-4}\Big(\tfrac{1}{2}w^2 - \tfrac{1}{8}(v_\epsilon v^\epsilon)^2 - 4v_\epsilon w^\epsilon + \tfrac{3}{2}v_\epsilon v^\epsilon w + \mathcal{O}(c^{-6})\Big\}. \end{aligned} \tag{200}$$

The Lagrangian given by Eq. (200) leads to the following equation of motion:

$$c\frac{d}{dy^0}\Big\{v^\alpha\Big(1+c^{-2}\big(3w-\tfrac{1}{2}v_\epsilon v^\epsilon\big) - 4c^{-2}w^\alpha + \mathcal{O}(c^{-4})\Big)\Big\} = -\partial^\alpha w\Big\{1 - c^{-2}\big(\tfrac{3}{2}v_\epsilon v^\epsilon + w\big)\Big\} \; - \; \frac{4}{c^2}v_\epsilon\partial^\alpha w^\epsilon + \mathcal{O}(c^{-4}). \tag{201}$$

It can be shown that Eq. (201) is equivalent to the geodesic equation for the local reference frame that is freely falling in the external gravitational field. However, Eq. (201) has the advantage as it allows to separate relativistic quantities and to study the motion of the system in a more straightforward way.

The condition that the test particle is to remain at rest with respect to the accelerating frame, then, amounts to demanding conservation of its three-dimensional momentum $p^\alpha = \partial L_{\rm tot}/\partial v^\alpha$:

$$p^\alpha = v^\alpha\left(1+c^{-2}\big(3w-\tfrac{1}{2}v_\epsilon v^\epsilon\big)\right) - 4c^{-2}w^\alpha + \mathcal{O}(c^{-4}). \tag{202}$$

In other words, the total time derivative of p^α must vanish. We assume that the observer is located at the spatial origin, $y^\alpha = 0$. Then this condition amounts to requiring that the first spatial derivatives $\partial_\beta w_b$ and $\partial_\beta w^\alpha_b$ are equal to that produced by the gravity of body a only, $g^a_{mn}(y_a)$. Thus, we require the following two conditions to hold:

$$\lim_{y_a\to 0}\frac{\partial}{\partial y^\beta_a}w(y_a) \;=\; \frac{\partial}{\partial y^\beta_a}w_a\Big|_{y_a=0} + \mathcal{O}(c^{-4}), \tag{203}$$

$$\lim_{y_a\to 0}\frac{\partial}{\partial y^\beta_a}w^\alpha(y_a) \;=\; \frac{\partial}{\partial y^\beta_a}w^\alpha_a\Big|_{y_a=0} + \mathcal{O}(c^{-2}). \tag{204}$$

We also require that on the world-line of the body its momentum (202) is to depend only on the potentials produced by the body itself and is independent on external gravitational fields, namely:

$$p^\alpha|_{y_a} = v^\alpha\left(1+c^{-2}\big(3w_a-\tfrac{1}{2}v_\epsilon v^\epsilon\big)\right) - 4c^{-2}w^\alpha_a + \mathcal{O}(c^{-4}). \tag{205}$$

Thus, we require that another pair of conditions hold:

$$\lim_{y_a\to 0} w(y_a) \;=\; w_a\Big|_{y_a=0} + \mathcal{O}(c^{-4}), \tag{206}$$

$$\lim_{y_a\to 0} w^\alpha(y_a) \;=\; w^\alpha_a\Big|_{y_a=0} + \mathcal{O}(c^{-2}). \tag{207}$$

Physically, we choose the coordinate system $\{y^m_a\}$ in the proper reference frame of the body a such that along the world-line of the body the external gravitational potentials in w_b and w^α_b, $b \neq a$ and the first spatial derivatives $\partial_\beta w_b$ and $\partial_\beta w^\alpha_b$ are absent, so that the spacetime of the N-body system in the local coordinates along the world-line $g_{mn}(y_a)$ reduces to the spacetime produced by the gravity of body a only, $g^a_{mn}(y_a)$. To put it another way, on the world-line of the body a, the presence of the external gravity will be completely compensated by the frame-reaction potentials. Consequently, in accordance with the Equivalence Principle, the external gravitational field in the body-centric metric will be manifested only via relativistic tidal effects in the vicinity of the world-line of body a. These requirements constitute a generalized set of Fermi-conditions in the case of the N-body system.

Therefore, summarizing Eqs. (203)–(204) and (206)–(207), we require that the following relations involving scalar and vector potentials w and w^α hold along the world-line:

$$\lim_{|y_a|\to 0} w(y_a) \;=\; w_a|_{y_a=0} + \mathcal{O}(c^{-4}), \qquad \lim_{|y_a|\to 0} \frac{\partial}{\partial y_a^\beta} w(y_a) = \frac{\partial}{\partial y_a^\beta} w_a\Big|_{y_a=0} + \mathcal{O}(c^{-4}), \tag{208}$$

$$\lim_{|y_a|\to 0} w^\alpha(y_a) \;=\; w_a^\alpha|_{y_a=0} + \mathcal{O}(c^{-2}), \qquad \lim_{|y_a|\to 0} \frac{\partial}{\partial y_a^\beta} w^\alpha(y_a) = \frac{\partial}{\partial y_a^\beta} w_a^\alpha\Big|_{y_a=0} + \mathcal{O}(c^{-2}). \tag{209}$$

Applying these conditions yields additional equations that allow us to determine uniquely the explicit form of the coordinate transformation functions $\mathcal{K}_a, \mathcal{L}_a$ and $\mathcal{Q}_a^\alpha$, allowing us to fix the time in the local coordinates. As it was demonstrated in [16], the conditions (208)–(209) can be used to determine frame-reaction potentials. These potentials are the combination of the terms on the right-hand sides of Eqs. (154)–(155) that depend only on the coordinate transformation functions $\mathcal{K}_a, \mathcal{L}_a$ and $\mathcal{Q}_a^\alpha$ and their derivatives. In addition, this procedure also enables us to determine the equations of motion of an observer with respect to the inertial reference system (logical, but somewhat unexpected result), written in his own coordinates $\{y_a^k\} \equiv (y_a^0, \mathbf{y}_a)$. These equations of motion will provide the dynamical conditions necessary to fix the remaining freedom in the coordinate transformation functions $\mathcal{K}_a, \mathcal{L}_a$ and $\mathcal{Q}_a^\alpha$.

B. Application of the dynamical conditions to fix the proper reference frame

Imposing the generalized Fermi conditions (208)–(209) on the potentials w and w^α, which are given by Eqs. (154)–(155), and denoting $\bar{u}$ to be the value of any function u on the world-line of body a via

$$\lim_{|y_a|\to 0} u(y_a) = u|_{y_a=0} = \bar{u}, \tag{210}$$

results in the following set of partial differential equations set on the world-line of the body a:

$$\sum_{b\neq a} \bar{w}_b - \frac{\partial \kappa_{a_0}}{\partial y_a^0} - \tfrac{1}{2} v_{a_0\epsilon} v_{a_0}^\epsilon - \frac{1}{c^2}\Big\{ \frac{\partial \mathcal{L}_a}{\partial y_a^0} + c v_{a_0\epsilon} \frac{\partial \mathcal{Q}_a^\epsilon}{\partial y_a^0} + \tfrac{1}{2}\Big(\frac{\partial \kappa_{a_0}}{\partial y_a^0}\Big)^2 - \Big(\frac{\partial \kappa_{a_0}}{\partial y_a^0} + \tfrac{1}{2} v_{a_0\epsilon} v_{a_0}^\epsilon\Big)^2 \Big\} \;=\; \mathcal{O}(c^{-4}), \tag{211}$$

$$\sum_{b\neq a} \frac{\partial \bar{w}_b}{\partial y_a^\beta} + a_{a_0\beta} - \frac{1}{c^2}\Big\{ \frac{\partial^2 \mathcal{L}_a}{\partial y_a^\beta \partial y_a^0} + c v_{a_0\epsilon} \frac{\partial^2 \mathcal{Q}_a^\epsilon}{\partial y_a^\beta \partial y_a^0} + a_{a_0\beta}\Big(\frac{\partial \kappa_{a_0}}{\partial y_a^0} + v_{a_0\epsilon} v_{a_0}^\epsilon\Big)\Big\} \;=\; \mathcal{O}(c^{-4}), \tag{212}$$

$$\sum_{b\neq a} \bar{w}_b^\alpha - \tfrac{1}{4}\Big\{ \gamma^{\alpha\epsilon} \frac{1}{c} \frac{\partial \mathcal{L}_a}{\partial y_a^\epsilon} + c \frac{\partial \mathcal{Q}_a^\alpha}{\partial y_a^0} + \gamma^{\alpha\epsilon} v_{a_0\lambda} \frac{\partial \mathcal{Q}_a^\lambda}{\partial y_a^\epsilon} - v_{a_0}^\alpha \frac{\partial \kappa_{a_0}}{\partial y_a^0} \Big\} \;=\; \mathcal{O}(c^{-2}), \tag{213}$$

$$\sum_{b\neq a} \frac{\partial \bar{w}_\alpha^b}{\partial y_a^\beta} - \tfrac{1}{4}\Big\{ \frac{1}{c} \frac{\partial^2 \mathcal{L}_a}{\partial y_a^\alpha \partial y_a^\beta} + c \gamma_{\alpha\lambda} \frac{\partial^2 \mathcal{Q}_a^\lambda}{\partial y_a^0 \partial y_a^\beta} + v_{a_0\lambda} \frac{\partial^2 \mathcal{Q}_a^\lambda}{\partial y_a^\alpha \partial y_a^\beta} + v_{a_0\alpha} a_{a_0\beta} \Big\} \;=\; \mathcal{O}(c^{-2}), \tag{214}$$

Equations (211)–(214) may be used to determine uniquely the transformation functions $\mathcal{K}_a, \mathcal{L}_a$ and $\mathcal{Q}_a^\alpha$. Indeed, from the first two equations above, (211) and (212), we immediately have:

$$\sum_{b\neq a} \bar{U}_b - \frac{\partial \kappa_{a_0}}{\partial y_a^0} - \tfrac{1}{2} v_{a_0\epsilon} v_{a_0}^\epsilon \;=\; \mathcal{O}(c^{-4}), \tag{215}$$

$$\sum_{b\neq a} \frac{\partial \bar{U}_b}{\partial y_a^\beta} + a_{a_0\beta} \;=\; \mathcal{O}(c^{-4}), \tag{216}$$

where we, for convenience, have split the scalar potential w_b into Newtonian (U_b) and post-Newtonian (δw_b) parts:

$$w_b = U_b + \frac{1}{c^2} \delta w_b + \mathcal{O}(c^{-4}). \tag{217}$$

This way, that gravitational scalar and vector potentials in the proper reference frame of the body a defined by Eqs. (218)–(219) have the following form as functions of local coordinates:

$$w_b(y_a) \;=\; U_b(x(y_a)) + \frac{1}{c^2}\Big(\delta w_b(x(y_a)) - 2(v_{a_0\epsilon} v_{a_0}^\epsilon) U_b(x(y_a)) + 4 v_{a_0\epsilon} w_b^\epsilon(x(y_a)) \Big) + \mathcal{O}(c^{-4}), \tag{218}$$

$$w_b^\alpha(y_a) \;=\; w_b^\alpha(x(y_a)) - v_{a_0}^\alpha U_b(x(y_a)) + \mathcal{O}(c^{-2}). \tag{219}$$

Equations (178) and (215) allow us to determine κ_{a_0} and present the solution for the function $\mathcal{K}_a$ as:

$$\mathcal{K}_a(y_a) = \int_{y^0_{a_0}}^{y^0_a}\Big(\sum_{b\neq a}\bar{U}_b - \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)dy'^0_a - c(v_{a_0\epsilon}y^\epsilon_a) + \mathcal{O}(c^{-4}). \tag{220}$$

From Eq. (216) we also have the well-known Newtonian equation of motion of the body a:

$$a^\alpha_{a_0} \;=\; -\gamma^{\alpha\epsilon}\sum_{b\neq a}\frac{\partial \bar{U}_b}{\partial y^\epsilon_a} + \mathcal{O}(c^{-4}). \tag{221}$$

Substituting the expression (215) into Eq. (198), we can determine the function $\mathcal{Q}^\alpha_a$:

$$\mathcal{Q}^\alpha_a(y_a) \;=\; q^\alpha_{a_0} - \Big(\tfrac{1}{2}v^\alpha_{a_0}v^\epsilon_{a_0} + \gamma^{\alpha\epsilon}\sum_{b\neq a}\bar{U}_b + \omega^{\alpha\epsilon}_{a_0}\Big)y_{a\epsilon} + a_{a_0\epsilon}\Big(y^\alpha_a y^\epsilon_a - \tfrac{1}{2}\gamma^{\alpha\epsilon}y_{a\lambda}y^\lambda_a\Big) + \mathcal{O}(c^{-2}). \tag{222}$$

Finally, the general solution for the function $\mathcal{L}_a$, given by Eq. (196), now takes the following form:

$$\mathcal{L}_a(y_a) = \ell_{a_0} + \ell_{a_0\lambda}\, y^\lambda_a + \tfrac{1}{2}\ell_{a_0\lambda\mu}\, y^\lambda_a y^\mu_a - \tfrac{1}{6}c\Big(\sum_{b\neq a}c\frac{\partial \bar{U}_b}{\partial y^0_a} - v_{a_0\epsilon}a^\epsilon_{a_0}\Big)(y_{a\nu}y^\nu_a) + \tfrac{1}{10}c(\dot{a}_{a_0\epsilon}y^\epsilon_a)(y_{a\nu}y^\nu_a) + \delta\ell_a(y_a) + \mathcal{O}(c^{-2}). \tag{223}$$

The next task is to find the remaining undetermined time-dependent functions present in $\mathcal{Q}^\alpha_a$ and $\mathcal{L}_a$, as given by Eqs. (222) and (223). To do this, we rewrite the remaining parts of Eqs. (211)–(214) as a system of partial differential equations, again set on the world-line of the body a:

$$\sum_{b\neq a}\delta\bar{w}_b - \Big\{\frac{\partial \mathcal{L}_a}{\partial y^0_a} + c v_{a_0\epsilon}\frac{\partial \mathcal{Q}^\epsilon_a}{\partial y^0_a} + \tfrac{1}{2}\Big(\frac{\partial \kappa_{a_0}}{\partial y^0_a}\Big)^2 - \Big(\frac{\partial \kappa_{a_0}}{\partial y^0_a} + \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)^2\Big\} \;=\; \mathcal{O}(c^{-2}), \tag{224}$$

$$\sum_{b\neq a}\frac{\partial}{\partial y^\beta_a}\delta\bar{w}_b - \Big\{\frac{\partial^2 \mathcal{L}_a}{\partial y^\beta_a\partial y^0_a} + c v_{a_0\epsilon}\frac{\partial^2 \mathcal{Q}^\epsilon_a}{\partial y^\beta_a\partial y^0_a} + a_{a_0\beta}\Big(\frac{\partial \kappa_{a_0}}{\partial y^0_a} + v_{a_0\epsilon}v^\epsilon_{a_0}\Big)\Big\} \;=\; \mathcal{O}(c^{-2}), \tag{225}$$

$$\sum_{b\neq a}\bar{w}^\alpha_b - \tfrac{1}{4}\Big\{\gamma^{\alpha\epsilon}\frac{1}{c}\frac{\partial \mathcal{L}_a}{\partial y^\epsilon_a} + c\frac{\partial \mathcal{Q}^\alpha_a}{\partial y^0_a} + \gamma^{\alpha\epsilon}v_{a_0\lambda}\frac{\partial \mathcal{Q}^\lambda_a}{\partial y^\epsilon_a} - v^\alpha_{a_0}\frac{\partial \kappa_{a_0}}{\partial y^0_a}\Big\} \;=\; \mathcal{O}(c^{-2}), \tag{226}$$

$$\sum_{b\neq a}\frac{\partial \bar{w}_{b\alpha}}{\partial y^\beta_a} - \tfrac{1}{4}\Big\{\frac{1}{c}\frac{\partial^2 \mathcal{L}_a}{\partial y^\alpha_a\partial y^\beta_a} + c\gamma_{\alpha\lambda}\frac{\partial^2 \mathcal{Q}^\lambda_a}{\partial y^0_a\partial y^\beta_a} + v_{a_0\lambda}\frac{\partial^2 \mathcal{Q}^\lambda_a}{\partial y^\alpha_a\partial y^\beta_a} + v_{a_0\alpha}a_{a_0\beta}\Big\} \;=\; \mathcal{O}(c^{-2}). \tag{227}$$

These equations may be used to determine the remaining unknown time-dependent functions ℓ_{a_0}, $\ell_{a_0\lambda}$, $\ell_{a_0\lambda\mu}$, and also $q^\alpha_{a_0}$ and $\omega^{\alpha\epsilon}_{a_0}$, which are still present in the transformation functions. Thus, substituting the solutions for $\mathcal{K}_a$ and $\mathcal{Q}^\alpha_a$, given respectively by Eqs. (220) and (222), into Eq. (224) leads to the following solution for $\dot{\ell}_{a_0}$:

$$\tfrac{1}{c}\dot{\ell}_{a_0} \;=\; -v_{a_0\epsilon}\dot{q}^\epsilon_{a_0} - \tfrac{1}{8}(v_{a_0\epsilon}v^\epsilon_{a_0})^2 - \tfrac{3}{2}(v_{a_0\epsilon}v^\epsilon_{a_0})\sum_{b\neq a}\bar{U}_b + 4v_{a_0\epsilon}\sum_{b\neq a}\bar{w}^\epsilon_b + \tfrac{1}{2}(\sum_{b\neq a}\bar{U}_b)^2 + \sum_{b\neq a}\delta\bar{w}_b + \mathcal{O}(c^{-2}). \tag{228}$$

Next, Eq. (225) results in the following equation for $\dot{\ell}^\alpha_{a_0}$:

$$\tfrac{1}{c}\dot{\ell}^\alpha_{a_0} = \sum_{b\neq a}\Big(\frac{\partial}{\partial y_{a\alpha}}\delta\bar{w}_b + 4v_{a_0\epsilon}\frac{\partial \bar{w}^\epsilon_b}{\partial y_{a\alpha}}\Big) - a^\alpha_{a_0}\sum_{b\neq a}\bar{U}_b + 2(v_{a_0\epsilon}v^\epsilon_{a_0})a^\alpha_{a_0} + \tfrac{1}{2}v^\alpha_{a_0}(v_{a_0\epsilon}a^\epsilon_{a_0}) + v^\alpha_{a_0}\sum_{b\neq a}c\frac{\partial \bar{U}_b}{\partial y^0_a} - v_{a_0\epsilon}\dot{\omega}^{\alpha\epsilon}_{a_0} + \mathcal{O}(c^{-2}). \tag{229}$$

From Eq. (226) we can determine $\ell^\alpha_{a_0}$:

$$\tfrac{1}{c}\ell^\alpha_{a_0} \;=\; -\dot{q}^\alpha_{a_0} + 4\sum_{b\neq a}\bar{w}^\alpha_b - 2v^\alpha_{a_0}\sum_{b\neq a}\bar{U}_b - v_{a_0\epsilon}\,\omega^{\alpha\epsilon}_{a_0} + \mathcal{O}(c^{-2}). \tag{230}$$

Eq. (227) leads to the following solution for $\ell^{\alpha\beta}_{a_0}$:

$$\tfrac{1}{c}\ell^{\alpha\beta}_{a_0} \;=\; 4\sum_{b\neq a}\frac{\partial \bar{w}^\alpha_b}{\partial y_{a\beta}} + \tfrac{4}{3}\gamma^{\alpha\beta}\sum_{b\neq a}c\frac{\partial \bar{U}_b}{\partial y^0_a} + \tfrac{2}{3}\gamma^{\alpha\beta}v_{a_0\epsilon}a^\epsilon_{a_0} + \tfrac{5}{2}v^\alpha_{a_0}a^\beta_{a_0} - \tfrac{1}{2}v^\beta_{a_0}a^\alpha_{a_0} + \dot{\omega}^{\alpha\beta}_{a_0} + \mathcal{O}(c^{-2}). \tag{231}$$

The quantity $\ell^{\alpha\beta}_{a_0}$ is an STF tensor. The expression on the right-hand side must, therefore, be also symmetric. This can be achieved by choosing the anti-symmetric tensor $\dot\omega^{\alpha\beta}_{a_0}$ appropriately. This can be done uniquely, resulting in

$$\dot\omega^{\alpha\beta}_{a_0} = -\tfrac{3}{2}(v^\alpha_{a_0}a^\beta_{a_0} - v^\beta_{a_0}a^\alpha_{a_0}) + 2\sum_{b\neq a}\Big(\frac{\partial\bar w^\beta_b}{\partial y_{a\alpha}} - \frac{\partial\bar w^\alpha_b}{\partial y_{a\beta}}\Big) + \mathcal{O}(c^{-2}), \tag{232}$$

which leads to the following solution for $\ell^{\alpha\beta}_{a_0}$:

$$\tfrac{1}{c}\ell^{\alpha\beta}_{a_0} \;=\; 2\sum_{b\neq a}\Big(\frac{\partial\bar w^\beta_b}{\partial y_{a\alpha}} + \frac{\partial\bar w^\alpha_b}{\partial y_{a\beta}}\Big) + \tfrac{4}{3}\gamma^{\alpha\beta}\sum_{b\neq a}c\frac{\partial\bar U_b}{\partial y^0_a} + \Big(\tfrac{2}{3}\gamma^{\alpha\beta}v_{a_0\epsilon}a^\epsilon_{a_0} + v^\alpha_{a_0}a^\beta_{a_0} + v^\beta_{a_0}a^\alpha_{a_0}\Big) + \mathcal{O}(c^{-2}). \tag{233}$$

Also, using Eq. (232), one can present a solution for $\dot\ell^\alpha_{a_0}$, Eq. (229), in the form

$$\tfrac{1}{c}\dot\ell^\alpha_{a_0} = \sum_{b\neq a}\frac{\partial\delta\bar w_b}{\partial y_{a\alpha}} + 2v_{a_0\epsilon}\sum_{b\neq a}\Big(\frac{\partial\bar w^\epsilon_b}{\partial y_{a\alpha}} + \frac{\partial\bar w^\alpha_b}{\partial y_{a\epsilon}}\Big) + \Big(\tfrac{1}{2}(v_{a_0\epsilon}v^\epsilon_{a_0}) - \sum_{b\neq a}\bar U_b\Big)a^\alpha_{a_0} + 2v^\alpha_{a_0}(a_{a_0\epsilon}v^\epsilon_{a_0}) + v^\alpha_{a_0}\sum_{b\neq a}c\frac{\partial\bar U_b}{\partial y^0_a} + \mathcal{O}(c^{-2}). \tag{234}$$

Finally, Eqs. (229) and (230) allow us to determine the equation for $q^\alpha_{a_0}$. Indeed, differentiating Eq. (230) with respect to time and subtracting the result from Eq. (229), we obtain the following equation for $\ddot q^\alpha_{a_0}$:

$$\begin{aligned}\ddot q^\alpha_{a_0} \;=\; & -\gamma^{\alpha\epsilon}\sum_{b\neq a}\frac{\partial\delta\bar w_b}{\partial y^\epsilon_a} + 4\sum_{b\neq a}c\frac{\partial\bar w^\alpha_b}{\partial y^0_a} - 4v_{a_0\epsilon}\sum_{b\neq a}\frac{\partial\bar w^\epsilon_b}{\partial y_{a\alpha}} - a^\alpha_{a_0}\sum_{b\neq a}\bar U_b - 3v^\alpha_{a_0}\sum_{b\neq a}c\frac{\partial\bar U_b}{\partial y^0_a} - \\ & - \Big(\tfrac{1}{2}v^\alpha_{a_0}v^\epsilon_{a_0} + \omega^{\alpha\epsilon}_{a_0}\Big)a_{a_0\epsilon} - 2a^\alpha_{a_0}(v_{a_0\epsilon}v^\epsilon_{a_0}) + \mathcal{O}(c^{-2}).\end{aligned} \tag{235}$$

Therefore, we can use Eq. (235) to completely determine the function q^α_0.

The true position of a body a includes terms to all orders, not just the first-order (Galilean) term. This led us to introduce the post-Galilean vector, $x^\alpha_{a_0}(y^0_a)$, defined by Eq. (39). Combining this definition with Eqs. (216) and (235), we can now relate the magnitude of the frame-reaction force (acting on a unit mass), written in the local reference frame of the body a, to the acceleration produced by the external gravitational field $\ddot x^\alpha_{a_0} = \ddot z^\alpha_{a_0} + c^{-2}\ddot q^\alpha_{a_0} + \mathcal{O}(c^{-4})$:

$$\begin{aligned}\ddot x^\alpha_{a_0} \;=\; & -\sum_{b\neq a}\frac{\partial\bar w_b}{\partial y^\epsilon_a}\Big\{\gamma^{\alpha\epsilon} - \frac{1}{c^2}\Big(\gamma^{\alpha\epsilon}\sum_{b\neq a}\bar U_b + \tfrac{1}{2}v^\alpha_{a_0}v^\epsilon_{a_0} + \omega^{\alpha\epsilon}_{a_0} + 2\gamma^{\alpha\epsilon}(v_{a_0\epsilon}v^\epsilon_{a_0})\Big)\Big\} + \\ & + \frac{1}{c^2}\Big(4\sum_{b\neq a}c\frac{\partial\bar w^\alpha_b}{\partial y^0_a} - 4v_{a_0\epsilon}\sum_{b\neq a}\frac{\partial\bar w^\epsilon_b}{\partial y_{a\alpha}} - 3v^\alpha_{a_0}\sum_{b\neq a}c\frac{\partial\bar U_b}{\partial y^0_a}\Big) + \mathcal{O}(c^{-4}).\end{aligned} \tag{236}$$

The equation of motion (236) establishes the correspondence between the external gravitational field and the fictitious frame-reaction acceleration $\ddot x^\alpha_{a_0}(y^0)$ that is needed to keep the body at rest in its proper reference frame. This frame-reaction force balances the effect of the external gravity acting on the observer co-moving with the body.

C. Summary of results for the direct transformations

In the preceding sections, we sought to write the general post-Galilean transformation from a global, inertial frame of reference to a local, accelerating frame in the form

$$x^0 \;=\; y^0_a + c^{-2}\mathcal{K}_a(y^k_a) + c^{-4}\mathcal{L}_a(y^k_a) + \mathcal{O}(c^{-6}), \tag{237}$$

$$x^\alpha \;=\; y^\alpha_a + z^\alpha_{a_0}(y^0_a) + c^{-2}\mathcal{Q}^\alpha_a(y^k_a) + \mathcal{O}(c^{-4}). \tag{238}$$

To determine the unknown functions $\mathcal{K}_a$, $\mathcal{Q}^\alpha_a$, and $\mathcal{L}_a$, we used the following approach: i) we imposed the harmonic gauge conditions on the metric tensors in the global and local reference frames; ii) we ensured that the non-inertial local frame is non-rotating; iii) we introduced a co-moving accelerating observer at rest with respect to its proper accelerating frame of a body a; iv) we required that a co-moving observers's ordinary three-dimensional linear momentum be conserved on the world-line occupied by the accelerating frame.

Together, these conditions were sufficient to determine $\mathcal{K}_a$, $\mathcal{Q}^\alpha_a$, and $\mathcal{L}_a$ unambiguously. $\mathcal{K}_a$ and $\mathcal{Q}^\alpha_a$ are given by:

$$\mathcal{K}_a(y_a) \;=\; \int_{y^0_{a_0}}^{y^0_a}\Big(\sum_{b\neq a}\bar U_b - \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)dy'^0_a - c(v_{a_0\epsilon}y^\epsilon_a) + \mathcal{O}(c^{-4}), \tag{239}$$

$$\mathcal{Q}^\alpha_a(y_a) \;=\; q^\alpha_{a_0} - \Big(\tfrac{1}{2}v^\alpha_{a_0}v^\epsilon_{a_0} + \gamma^{\alpha\epsilon}\sum_{b\neq a}\bar U_b + \omega^{\alpha\epsilon}_{a_0}\Big)y_{a\epsilon} + a_{a_0\epsilon}\Big(y^\alpha_a y^\epsilon_a - \tfrac{1}{2}\gamma^{\alpha\epsilon}y_{a\lambda}y^\lambda_a\Big) + \mathcal{O}(c^{-2}), \tag{240}$$

with the anti-symmetric relativistic precession matrix $\omega^{\alpha\beta}_{a_0}$ given by Eq. (232) and the post-Newtonian component of the spatial coordinate in the local frame, $q^\alpha_{a_0}$, given by Eq. (235).

To present the last transformation function, $\mathcal{L}_a(y_a)$, we use the explicit dependence of the Newtonian and post-Newtonian components of scalar and vector potentials on the body-centric potentials, as given by Eqs. (156) and (157). As a result, $\mathcal{L}_a(y_a)$ is given by:

$$\mathcal{L}_a(y_a) \;=\; \ell_{a_0} + \ell_{a_0\lambda}\,y^\lambda_a + \tfrac{1}{2}\Big(\ell_{a_0\lambda\mu} + \tfrac{1}{3}c\gamma_{\lambda\mu}\big(v_{a_0\epsilon}a^\epsilon_{a_0} - \sum_{b\neq a}\dot{\bar U}_b\big)\Big)y^\lambda_a y^\mu_a + \tfrac{1}{10}c(\dot a_{a_0\epsilon}y^\epsilon_a)(y_{a\nu}y^\nu_a) + \delta\ell_a(y_a) + \mathcal{O}(c^{-2}), \tag{241}$$

where the functions ℓ_{a_0}, $\ell^\alpha_{a_0}$ and $\ell^{\alpha\beta}_{a_0}$ are defined using the following expressions:

$$\tfrac{1}{c}\dot\ell_{a_0} \;=\; -v_{a_0\epsilon}\dot q^\epsilon_{a_0} - \tfrac{1}{8}(v_{a_0\epsilon}v^\epsilon_{a_0})^2 - \tfrac{3}{2}(v_{a_0\epsilon}v^\epsilon_{a_0})\sum_{b\neq a}\bar U_b + 4v_{a_0\epsilon}\sum_{b\neq a}\bar w^\epsilon_b + \tfrac{1}{2}(\sum_{b\neq a}\bar U_b)^2 + \sum_{b\neq a}\delta\bar w_b + \mathcal{O}(c^{-2}), \tag{242}$$

$$\tfrac{1}{c}\ell^\alpha_{a_0} \;=\; -\dot q^\alpha_{a_0} + 4\sum_{b\neq a}\bar w^\alpha_b - 2v^\alpha_{a_0}\sum_{b\neq a}\bar U_b - v_{a_0\epsilon}\,\omega^{\alpha\epsilon}_{a_0} + \mathcal{O}(c^{-2}), \tag{243}$$

$$\tfrac{1}{c}\ell^{\alpha\beta}_{a_0} \;=\; 2\sum_{b\neq a}\Big(\frac{\partial\bar w^\beta_b}{\partial y_{a\alpha}} + \frac{\partial\bar w^\alpha_b}{\partial y_{a\beta}}\Big) + \tfrac{4}{3}\gamma^{\alpha\beta}\sum_{b\neq a}c\frac{\partial\bar U_b}{\partial y^0_a} + \Big(\tfrac{2}{3}\gamma^{\alpha\beta}v_{a_0\epsilon}a^\epsilon_{a_0} + v^\alpha_{a_0}a^\beta_{a_0} + v^\beta_{a_0}a^\alpha_{a_0}\Big) + \mathcal{O}(c^{-2}). \tag{244}$$

Substituting these solutions for the functions $\mathcal{K}_a$, $\mathcal{Q}^\alpha_a$ and $\mathcal{L}_a$ into the expressions for the potentials w and w^α given by Eqs. (154)–(155), we find the following form for these potentials:

$$\begin{aligned} w(y_a) \;=\; & \sum_b w_b(y_a) - \sum_{b\neq a}\Big(\bar w_b + y^\epsilon_a\frac{\partial\bar w_b}{\partial y^\epsilon_a}\Big) - \\ & - \frac{1}{c^2}\Big\{\tfrac{1}{2}y^\epsilon_a y^\lambda_a\Big[\gamma_{\epsilon\lambda}a_{a_0\mu}a^\mu_{a_0} + a_{a_0\epsilon}a_{a_0\lambda} + 2\dot a_{a_0\epsilon}v_{a_0\lambda} + 2v_{a_0\epsilon}\dot a_{a_0\lambda} + \gamma_{\epsilon\lambda}\sum_{b\neq a}\ddot{\bar U}_b + \\ & + 2\sum_{b\neq a}\Big(\frac{\partial\dot{\bar w}_{b\lambda}}{\partial y^\epsilon_a} + \frac{\partial\dot{\bar w}_{b\epsilon}}{\partial y^\lambda_a}\Big)\Big] + \tfrac{1}{10}(\ddot a_{a_0\epsilon}y^\epsilon_a)(y_{a\mu}y^\mu_a) + \partial_0\delta\ell_a\Big\} + \mathcal{O}(c^{-4}), \end{aligned} \tag{245}$$

$$w^\alpha(y_a) \;=\; \sum_b w^\alpha_b(y_a) - \sum_{b\neq a}\Big(\bar w^\alpha_b + y^\epsilon_a\frac{\partial\bar w^\alpha_b}{\partial y^\epsilon_a}\Big) - \tfrac{1}{10}\big\{3y^\alpha_a y^\epsilon_a - \gamma^{\alpha\epsilon}y_{a\mu}y^\mu_a\big\}\dot a_{a_0\epsilon} - \tfrac{1}{4}\tfrac{1}{c}\frac{\partial}{\partial y_{a\alpha}}\delta\ell_a + \mathcal{O}(c^{-2}). \tag{246}$$

Substitution of these solutions for the potentials into the expressions for the metric tensor given by Eqs. (151)–(153) leads to the following form of the local metric tensor:

$$\begin{aligned} g_{00}(y_a) \;=\; & 1 - \frac{2}{c^2}\Big\{\sum_b w_b(y_a) - \sum_{b\neq a}\Big(\bar w_b + y^\epsilon_a\frac{\partial\bar w_b}{\partial y^\epsilon_a}\Big)\Big\} + \frac{2}{c^4}\Big\{\Big[\sum_b U_b(y_a) - \sum_{b\neq a}\Big(\bar U_b + y^\epsilon_a\frac{\partial\bar U_b}{\partial y^\epsilon_a}\Big)\Big]^2 + \\ & + \tfrac{1}{2}y^\epsilon_a y^\lambda_a\Big[\gamma_{\epsilon\lambda}a_{a_0\mu}a^\mu_{a_0} + a_{a_0\epsilon}a_{a_0\lambda} + 2\dot a_{a_0\epsilon}v_{a_0\lambda} + 2v_{a_0\epsilon}\dot a_{a_0\lambda} + \gamma_{\epsilon\lambda}\sum_{b\neq a}\ddot{\bar U}_b + \\ & + 2\sum_{b\neq a}\Big(\frac{\partial\dot{\bar w}_{b\lambda}}{\partial y^\epsilon_a} + \frac{\partial\dot{\bar w}_{b\epsilon}}{\partial y^\lambda_a}\Big)\Big] + \tfrac{1}{10}(\ddot a_{a_0\epsilon}y^\epsilon_a)(y_{a\mu}y^\mu_a) + \partial_0\delta\ell_a\Big\} + \mathcal{O}(c^{-6}), \end{aligned} \tag{247}$$

$$g_{0\alpha}(y_a) \;=\; -\gamma_{\alpha\lambda}\frac{4}{c^3}\Big\{\sum_b w^\lambda_b(y_a) - \sum_{b\neq a}\Big(\bar w^\lambda_b + y^\epsilon_a\frac{\partial\bar w^\lambda_b}{\partial y^\epsilon_a}\Big) - \tfrac{1}{10}\big\{3y^\lambda_a y^\epsilon_a - \gamma^{\lambda\epsilon}y_{a\mu}y^\mu_a\big\}\dot a_{a_0\epsilon}\Big\} + \frac{1}{c^4}\frac{\partial}{\partial y^\alpha_a}\delta\ell_a + \mathcal{O}(c^{-5}), \tag{248}$$

$$g_{\alpha\beta}(y_a) \;=\; \gamma_{\alpha\beta} + \gamma_{\alpha\beta}\frac{2}{c^2}\Big\{\sum_b w_b(y_a) - \sum_{b\neq a}\Big(\bar w_b + y^\epsilon_a\frac{\partial\bar w_b}{\partial y^\epsilon_a}\Big)\Big\} + \mathcal{O}(c^{-4}). \tag{249}$$

All terms in this metric are determined except for the function $\delta\ell_a$, which remains unknown. We note that the potentials Eqs. (245)–(246) depend on partial derivatives of $\delta\ell_a$. The same partial derivatives appear in the temporal and mixed components of the metric (247)–(248). The presence of these terms in the metric amounts to adding a

full time derivative to the Lagrangian that describes the system of the moving observer. Indeed, separating the terms that depend on $\delta\ell_a$ from the Lagrangian constructed with Eqs. (247)–(249), we get:

$$\delta L_{\delta\ell_a}=\frac{2}{c^4}\Big\{\frac{\partial\delta\ell_a}{\partial y^0_a}+\frac{v^\epsilon}{c}\frac{\partial\delta\ell_a}{\partial y^\epsilon_a}\Big\}+\mathcal{O}(c^{-6})=\frac{2}{c^4}\frac{d\delta\ell_a}{dy^0_a}+\mathcal{O}(c^{-6}). \tag{250}$$

As a result, the remainder of the gauge transformation depending on $\delta\ell_a$ will not change the dynamics in the system and, thus, it can be omitted and, so that the potentials w and w^α take the form:

$$\begin{aligned}
w(y_a)\ &=\ \sum_b w_b(y_a)-\sum_{b\not=a}\Big(\bar w_b+y^\epsilon_a\frac{\partial\bar w_b}{\partial y^\epsilon_a}\Big)-\\
&\quad -\frac{1}{c^2}\Big\{\tfrac{1}{2}y^\epsilon_a y^\lambda_a\Big[\gamma_{\epsilon\lambda}a_{a_0\mu}a^\mu_{a_0}+a_{a_0\epsilon}a_{a_0\lambda}+2\dot a_{a_0\epsilon}v_{a_0\lambda}+2v_{a_0\epsilon}\dot a_{a_0\lambda}+\gamma_{\epsilon\lambda}\sum_{b\not=a}\ddot{\bar U}_b+\\
&\quad +2\sum_{b\not=a}\Big(\frac{\partial\dot{\bar w}_{b\lambda}}{\partial y^\epsilon_a}+\frac{\partial\dot{\bar w}_{b\epsilon}}{\partial y^\lambda_a}\Big)\Big]+\tfrac{1}{10}(\ddot a_{a_0\epsilon}y^\epsilon_a)(y_{a\mu}y^\mu_a)\Big\}+\mathcal{O}(c^{-4}),
\end{aligned} \tag{251}$$

$$w^\alpha(y_a)\ =\ \sum_b w^\alpha_b(y_a)-\sum_{b\not=a}\Big(\bar w^\alpha_b+y^\epsilon_a\frac{\partial\bar w^\alpha_b}{\partial y^\epsilon_a}\Big)-\tfrac{1}{10}\big\{3y^\alpha_a y^\epsilon_a-\gamma^{\alpha\epsilon}y_{a\mu}y^\mu_a\big\}\dot a_{a_0\epsilon}+\mathcal{O}(c^{-2}). \tag{252}$$

We can now present the local metric in the following final form:

$$\begin{aligned}
g_{00}(y_a)\ &=\ 1-\frac{2}{c^2}\Big\{\sum_b w_b(y_a)-\sum_{b\not=a}\Big(\bar w_b+y^\epsilon_a\frac{\partial\bar w_b}{\partial y^\epsilon_a}\Big)\Big\}+\frac{2}{c^4}\Big\{\Big[\sum_b U_b(y_a)-\sum_{b\not=a}\Big(\bar U_b+y^\epsilon_a\frac{\partial\bar U_b}{\partial y^\epsilon_a}\Big)\Big]^2+\\
&\quad +\tfrac{1}{2}y^\epsilon_a y^\lambda_a\Big[\gamma_{\epsilon\lambda}a_{a_0\mu}a^\mu_{a_0}+a_{a_0\epsilon}a_{a_0\lambda}+2\dot a_{a_0\epsilon}v_{a_0\lambda}+2v_{a_0\epsilon}\dot a_{a_0\lambda}+\gamma_{\epsilon\lambda}\sum_{b\not=a}\ddot{\bar U}_b+2\sum_{b\not=a}\Big(\frac{\partial\dot{\bar w}_{b\lambda}}{\partial y^\epsilon_a}+\frac{\partial\dot{\bar w}_{b\epsilon}}{\partial y^\lambda_a}\Big)\Big]+\\
&\quad +\tfrac{1}{10}(\ddot a_{a_0\epsilon}y^\epsilon_a)(y_{a\mu}y^\mu_a)\Big\}+\mathcal{O}(c^{-6}),
\end{aligned} \tag{253}$$

$$g_{0\alpha}(y_a)\ =\ -\gamma_{\alpha\lambda}\frac{4}{c^3}\Big\{\sum_b w^\lambda_b(y_a)-\sum_{b\not=a}\Big(\bar w^\lambda_b+y^\epsilon_a\frac{\partial\bar w^\lambda_b}{\partial y^\epsilon_a}\Big)-\tfrac{1}{10}\big\{3y^\lambda_a y^\epsilon_a-\gamma^{\lambda\epsilon}y_{a\mu}y^\mu_a\big\}\dot a_{a_0\epsilon}\Big\}+\mathcal{O}(c^{-5}), \tag{254}$$

$$g_{\alpha\beta}(y_a)\ =\ \gamma_{\alpha\beta}+\gamma_{\alpha\beta}\frac{2}{c^2}\Big\{\sum_b w_b(y_a)-\sum_{b\not=a}\Big(\bar w_b+y^\epsilon_a\frac{\partial\bar w_b}{\partial y^\epsilon_a}\Big)\Big\}+\mathcal{O}(c^{-4}), \tag{255}$$

with the equation of motion of the body a, $\ddot x^\alpha_{a_0}=\ddot z^\alpha_{a_0}+c^{-2}\ddot q^\alpha_{a_0}+\mathcal{O}(c^{-4})$ given by Eq. (236).

The coordinate transformations that place an observer into this reference frame are given below:

$$\begin{aligned}
x^0\ &=\ y^0_a+c^{-2}\Big\{\int_{y^0_{a_0}}^{y^0_a}\Big(\sum_{b\not=a}\bar w_b-\tfrac{1}{2}(v_{a_0\epsilon}+c^{-2}\dot q_{a_0\epsilon})(v^\epsilon_{a_0}+c^{-2}\dot q^\epsilon_{a_0})\Big)dy'^0_a-c(v_{a_0\epsilon}+c^{-2}\dot q_{a_0\epsilon})y^\epsilon_a\Big\}+\\
&\quad +c^{-4}\Big\{\int_{y^0_{a_0}}^{y^0_a}\Big(-\tfrac{1}{8}(v_{a_0\epsilon}v^\epsilon_{a_0})^2-\tfrac{3}{2}(v_{a_0\epsilon}v^\epsilon_{a_0})\sum_{b\not=a}\bar U_b+4v_{a_0\epsilon}\sum_{b\not=a}\bar w^\epsilon_b+\tfrac{1}{2}\big(\sum_{b\not=a}\bar U_b\big)^2\Big)dy'^0_a+\\
&\quad +c\Big(4\sum_{b\not=a}\bar w_{b\epsilon}-2v_{a_0\epsilon}\sum_{b\not=a}\bar U_b-v^\lambda_{a_0}\,\omega_{a_0\epsilon\lambda}\Big)y^\epsilon_a+\tfrac{1}{2}c\Big[\gamma_{\lambda\mu}v_{a_0\epsilon}a^\epsilon_{a_0}+v_{a_0\lambda}a_{a_0\mu}+v_{a_0\mu}a_{a_0\lambda}+\\
&\quad +\gamma_{\lambda\mu}\sum_{b\not=a}\dot{\bar U}_b+2\sum_{b\not=a}\Big(\frac{\partial\bar w_{b\mu}}{\partial y^\lambda_a}+\frac{\partial\bar w_{b\lambda}}{\partial y^\mu_a}\Big)\Big]y^\lambda_a y^\mu_a+\tfrac{1}{10}c(\dot a_{a_0\epsilon}y^\epsilon_a)(y_{a\nu}y^\nu_a)\Big\}+\mathcal{O}(c^{-6}),
\end{aligned} \tag{256}$$

$$x^\alpha\ =\ y^\alpha_a+z^\alpha_{a_0}+c^{-2}\Big\{q^\alpha_{a_0}-\Big(\tfrac{1}{2}v^\alpha_{a_0}v^\epsilon_{a_0}+\gamma^{\alpha\epsilon}\sum_{b\not=a}\bar U_b+\omega^{\alpha\epsilon}_{a_0}\Big)y_{a\epsilon}+a_{a_0\epsilon}\Big(y^\alpha_a y^\epsilon_a-\tfrac{1}{2}\gamma^{\alpha\epsilon}y_{a\lambda}y^\lambda_a\Big)\Big\}+\mathcal{O}(c^{-4}). \tag{257}$$

The expressions (253)–(255) represent the harmonic metric tensor in the local coordinates of the accelerating reference frame. The form of the metric (253)–(255) and the coordinate transformations (256)–(257) are new and extend previous formulations (discussed, for instance, in [11]) obtained with different methods. These results can be used to develop models for high precision navigation and gravitational experiments. However, for a complete

description of these experiments we would need to establish the inverse coordinate transformations – the task that will be performed in the next section.

Finally, we note that the results presented here can be generalized to extended bodies. After a tedious but straightforward calculation (which can be done in a manner similar to [25]), we find that, given the accuracy of the measurements in the solar system, only the following two modifications need to be made to our solution:

$$\mathcal{K}_a(y_a) \;=\; \int_{y^0_{a_0}}^{y^0_a}\Big(\sum_{b\neq a}\langle U_b\rangle_a - \tfrac{1}{2}v_{a_0\epsilon}v^{\epsilon}_{a_0}\Big)dy'^0_a - c(v_{a_0\epsilon}y^{\epsilon}_a) + \mathcal{O}(c^{-4}), \tag{258}$$

$$a^{\alpha}_{a_0} \;=\; -\sum_{b\neq a}\langle\partial^{\alpha}U_b\rangle_a + \mathcal{O}(c^{-4}). \tag{259}$$

The difference between expressions for $\mathcal{K}_a(y_a)$ given by Eqs. (220) and (258) and also between those derived for $a^{\alpha}_{a_0}$ and given by Eqs. (221) and (259) is in the fact that the new expressions account for the interaction of the body's gravitational multipole moments with the background gravitational field, via the procedure of averaging the external gravitational field over the body's volume in the form of

$$\langle U_b\rangle_a = m_a^{-1}\int_a d^3y'_a\sigma U_b, \qquad \langle\partial^{\alpha}U_b\rangle_a = m_a^{-1}\int_a d^3y'_a\sigma\partial^{\alpha}U_b, \qquad m_a = \int_a d^3y'_a\sigma, \tag{260}$$

as opposed to accounting only for the value of that field on the body's world-line (via the limiting procedure defined by Eq. (210) taken for $\bar{U}_b$ and $\partial^{\alpha}\bar{U}_b$).

V. INVERSE TRANSFORMATIONS

In the preceding sections, we constructed an explicit form of the direct transformation between a global inertial and local accelerating reference frames by applying the harmonic gauge and dynamical conditions on the metric. Given the Jacobian matrix Eqs. (42)–(43), it was most convenient to work with the covariant form of the metric tensor, which could be expressed in terms of the accelerating coordinates by trivial application of the tensorial transformation rules Eq. (125).

The same logic suggests that if we were to work on the inverse transformation: that is, when it is the inverse Jacobian matrix $\{\partial y^m/\partial x^n\}$ that is given in explicit form, it is more convenient to work with the contravariant form of the metric tensor, to which this Jacobian can be applied readily. This simple observation leads us to the idea that we can get the inverse transformations—i.e., from the accelerated to the inertial frame—by simply repeating the previous calculations, but with the contravariant form of the metric tensor instead of the covariant form.

In this section, we show that this is indeed feasible, and accomplish something not usually found in the literature: construction of a method that can be applied for both direct and inverse transformations between inertial and accelerating reference frames at the same time, in a self-consistent manner.

A. The contravariant metric tensor in the local frame

We have obtained the solution of the gravitational field equations for the N-body problem, which in the barycentric reference frame has the form Eqs. (116)–(118) and (119), (120). It follows from Eq. (51) that the solution in the local frame could be in the following form:

$$g^{mn}(y_a) = \frac{\partial y^m_a}{\partial x^k}\frac{\partial y^n_a}{\partial x^l}\; g^{kl}(x(y_a)). \tag{261}$$

Using the coordinate transformations given by Eqs. (40)–(41) together with Eqs. (49)–(50), we determine the form of this metric tensor in the local reference frame associated with the body a:

$$\begin{aligned} g^{00}(y_a) \;=\; 1 &+ \frac{2}{c^2}\Big\{\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0} + \tfrac{1}{2}\gamma^{\epsilon\lambda}\frac{1}{c}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^{\epsilon}}\frac{1}{c}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^{\lambda}}\Big\} + \\ &+ \frac{2}{c^4}\Big\{\frac{\partial\hat{\mathcal{L}}_a}{\partial x^0} + \tfrac{1}{2}\Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}\Big)^2 + \gamma^{\epsilon\lambda}\frac{1}{c}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^{\epsilon}}\frac{1}{c}\frac{\partial\hat{\mathcal{L}}_a}{\partial x^{\lambda}} - \Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0} + \tfrac{1}{2}\gamma^{\epsilon\lambda}\frac{1}{c}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^{\epsilon}}\frac{1}{c}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^{\lambda}}\Big)^2\Big\} + \\ &+ \frac{2}{c^2}\sum_b\Big\{\Big(1 - \frac{2}{c^2}v_{a_0\epsilon}v^{\epsilon}_{a_0}\Big)w_b(x(y_a)) + \frac{4}{c^2}v_{a_0\epsilon}w^{\epsilon}_b(x(y_a))\Big\} + \end{aligned}$$

$$+\ \frac{2}{c^4}\Big(\sum_b w_b(x(y_a)) + \frac{\partial\hat{\mathcal{K}}_a}{\partial x^0} + \tfrac{1}{2}\gamma^{\epsilon\lambda}\frac{1}{c}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^\epsilon}\frac{1}{c}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^\lambda}\Big)^2 + \mathcal{O}(c^{-6}), \tag{262}$$

$$\begin{aligned} g^{0\alpha}(y_a) &= \frac{1}{c}\Big(\gamma^{\alpha\epsilon}\frac{1}{c}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^\epsilon} - v^\alpha_{a_0}\Big) + \frac{4}{c^3}\sum_b \Big(w^\alpha_b(x(y_a)) - v^\alpha_{a_0}w_b(x(y_a))\Big) + \\ &+ \frac{1}{c^3}\Big\{\gamma^{\alpha\epsilon}\frac{1}{c}\frac{\partial\hat{\mathcal{L}}_a}{\partial x^\epsilon} + c\frac{\partial\hat{\mathcal{Q}}^\alpha_a}{\partial x^0} + \gamma^{\epsilon\lambda}\frac{1}{c}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^\epsilon}\frac{\partial\hat{\mathcal{Q}}^\alpha_a}{\partial x^\lambda} - v^\alpha_{a_0}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}\Big\} + \mathcal{O}(c^{-5}), \end{aligned} \tag{263}$$

$$g^{\alpha\beta}(y_a) = \gamma^{\alpha\beta} + \frac{1}{c^2}\Big\{v^\alpha_{a_0}v^\beta_{a_0} + \gamma^{\alpha\lambda}\frac{\partial\hat{\mathcal{Q}}^\beta_a}{\partial x^\lambda} + \gamma^{\beta\lambda}\frac{\partial\hat{\mathcal{Q}}^\alpha_a}{\partial x^\lambda}\Big\} - \gamma^{\alpha\beta}\frac{2}{c^2}\sum_b w_b(x(y_a)) + \mathcal{O}(c^{-4}). \tag{264}$$

As in the case of the direct transformation, we impose the harmonic gauge condition on the metric, to help us establish explicit forms of the transformation functions $\hat{\mathcal{K}}_a$, $\hat{\mathcal{L}}_a$, and $\hat{\mathcal{Q}}^\alpha_a$.

Using Eqs. (80) and (82) to constrain the form of the metric tensor, given by Eqs. (262)–(264) in the local frame, we obtain:

$$\begin{aligned} g^{00}(y_a) &= 1 + \frac{2}{c^2}\Big\{\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0} + \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big\} + \frac{2}{c^4}\Big\{\frac{\partial\hat{\mathcal{L}}_a}{\partial x^0} + \tfrac{1}{2}\Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}\Big)^2 + \frac{v^\epsilon_{a_0}}{c}\frac{\partial\hat{\mathcal{L}}_a}{\partial x^\epsilon} - \Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0} + \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)^2\Big\} + \\ &+ \frac{2}{c^2}\sum_b\Big\{\Big(1 - \frac{2}{c^2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)w_b(x(y_a)) + \frac{4}{c^2}v_{a_0\epsilon}w^\epsilon_b(x(y_a))\Big\} + \\ &+ \frac{2}{c^4}\Big(\sum_b w_b(x(y_a)) + \frac{\partial\hat{\mathcal{K}}_a}{\partial x^0} + \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)^2 + \mathcal{O}(c^{-6}), \end{aligned} \tag{265}$$

$$g^{0\alpha}(y_a) = \frac{4}{c^3}\sum_b\Big(w^\alpha_b(x(y_a)) - v^\alpha_{a_0}w_b(x(y_a))\Big) + \frac{1}{c^3}\Big\{\gamma^{\alpha\epsilon}\frac{1}{c}\frac{\partial\hat{\mathcal{L}}_a}{\partial x^\epsilon} + c\frac{\partial\hat{\mathcal{Q}}^\alpha_a}{\partial x^0} + v^\epsilon_{a_0}\frac{\partial\hat{\mathcal{Q}}^\alpha_a}{\partial x^\epsilon} - v^\alpha_{a_0}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}\Big\} + \mathcal{O}(c^{-5}), \tag{266}$$

$$g^{\alpha\beta}(y_a) = \gamma^{\alpha\beta} + \frac{1}{c^2}\Big\{v^\alpha_{a_0}v^\beta_{a_0} + \gamma^{\alpha\lambda}\frac{\partial\hat{\mathcal{Q}}^\beta_a}{\partial x^\lambda} + \gamma^{\beta\lambda}\frac{\partial\hat{\mathcal{Q}}^\alpha_a}{\partial x^\lambda}\Big\} - \gamma^{\alpha\beta}\frac{2}{c^2}\sum_b w_b(x(y_a)) + \mathcal{O}(c^{-4}). \tag{267}$$

In analogy with Eq. (96), we determine the source term, $S^{mn}(y_a)$, for the N-body problem in the local reference frame as:

$$S^{mn}(y_a) = \frac{\partial y^m_a}{\partial x^k}\frac{\partial y^n_a}{\partial x^l}\ S^{kl}_b(x(y_a)), \tag{268}$$

where $S^{kl}_b(x(y_a)$ is the source term in the global frame. Thus, $S^{mn}(y_a)$ takes the form:

$$S^{00}(y_a) = \frac{1}{2}c^2\sum_b\Big\{\Big(1 - \frac{2}{c^2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)\sigma_b(x(y_a)) + \frac{4}{c^2}v_{a_0\epsilon}\sigma^\epsilon_b(x(y_a)) + \frac{2}{c^2}\Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0} + \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)\sigma_b(x(y_a)) + \mathcal{O}(c^{-4})\Big\}, \tag{269}$$

$$S^{0\alpha}(y_a) = c\sum_b\Big\{\sigma^\alpha_b(x(y_a)) - v^\alpha_{a_0}\sigma_b(x(y_a)) + \mathcal{O}(c^{-2})\Big\}, \tag{270}$$

$$S^{\alpha\beta}(y_a) = -\gamma^{\alpha\beta}\frac{1}{2}c^2\sum_b\Big\{\sigma_b(x(y_a)) + \mathcal{O}(c^{-2})\Big\}. \tag{271}$$

The metric tensor (7)–(9) and the partial-differential form of the gauge conditions (73)–(74) together with condition (82) allow one to simplify the expressions for the Ricci tensor and to present its contravariant components $R^{mn} = g^{mk}g^{nl}R_{kl}$ in the following form:

$$R^{00}(y_a) = \frac{1}{2}\Box_{y_a}\Big(c^{-2}g^{[2]00} + c^{-4}\Big\{g^{[4]00} - \frac{1}{2}\big(g^{[2]00}\big)^2\Big\}\Big) + \mathcal{O}(c^{-6}), \tag{272}$$

$$R^{0\alpha}(y_a) = c^{-3}\frac{1}{2}\Delta_{y_a}g^{[3]0\alpha} + \mathcal{O}(c^{-5}), \tag{273}$$

$$R^{\alpha\beta}(y_a) = c^{-2}\frac{1}{2}\Delta_{y_a}g^{[2]\alpha\beta} + \mathcal{O}(c^{-4}). \tag{274}$$

These expressions allow one to express the gravitational field equations as below:

$$\Box_{y_a}\Big[\sum_b\Big\{\Big(1 - \frac{2}{c^2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)w_b(x(y_a))\ +\ \frac{4}{c^2}v_{a_0\epsilon}w^\epsilon_b(x(y_a))\Big\} +$$

$$+\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}+\frac{1}{c^2}\Big\{\frac{\partial\hat{\mathcal{L}}_a}{\partial x^0}\;+\;\tfrac{1}{2}\Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}\Big)^2+\frac{v^\epsilon_{a_0}}{c}\frac{\partial\hat{\mathcal{L}}_a}{\partial x^\epsilon}-\Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)^2\Big\}+\mathcal{O}(c^{-4})\Big]=$$

$$=4\pi G\sum_b\Big\{\Big(1-\frac{2}{c^2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)\sigma_b(x(y_a))\;+\;\frac{4}{c^2}v_{a_0\epsilon}\sigma^\epsilon_b(x(y_a))+\frac{2}{c^2}\big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\big)\sigma_b(x(y_a))+\mathcal{O}(c^{-4})\Big\},\qquad(275)$$

$$\Delta_{y_a}\Big[\sum_b\Big(w^\alpha_b(x(y_a))-v^\alpha_{a_0}w_b(x(y_a))\Big)\;+\;\frac{1}{4}\Big\{\gamma^{\alpha\epsilon}\frac{1}{c}\frac{\partial\hat{\mathcal{L}}_a}{\partial x^\epsilon}+c\frac{\partial\hat{\mathcal{Q}}^\alpha_a}{\partial x^0}+v^\epsilon_{a_0}\frac{\partial\hat{\mathcal{Q}}^\alpha_a}{\partial x^\epsilon}-v^\alpha_{a_0}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}\Big\}+\mathcal{O}(c^{-2})\Big]=$$

$$=\;4\pi G\sum_b\Big\{\sigma^\alpha_b(x(y_a))-v^\alpha_{a_0}\sigma_b(x(y_a))\Big\}+\mathcal{O}(c^{-2}),\qquad(276)$$

$$\Delta_{y_a}\Big[\gamma^{\alpha\beta}\sum_b w_b(x(y_a))-\frac{1}{2}\Big(v^\alpha_{a_0}v^\beta_{a_0}\;+\;\gamma^{\alpha\lambda}\frac{\partial\hat{\mathcal{Q}}^\beta_a}{\partial x^\lambda}+\gamma^{\beta\lambda}\frac{\partial\hat{\mathcal{Q}}^\alpha_a}{\partial x^\lambda}\Big)+\mathcal{O}(c^{-2})\Big]=4\pi G\gamma^{\alpha\beta}\sum_b\sigma_b(x(y_a))+\mathcal{O}(c^{-2}).\qquad(277)$$

Remembering Eq. (150) and noting that $\Delta_{y_a}=\Delta_x+\mathcal{O}(c^{-2})$, with help of Eqs. (86)–(88), we can verify that Eqs. (275)–(277) are satisfied.

We now note that the Jacobian matrix on the right hand side on Eq. (261) is composed of the quantities $\partial y^m_a/\partial x^k$ that are functions of $\{x^k\}$, the same is true for $g^{kl}(x(y_a))$, which may also be treated only as a function of $\{x^k\}$. Therefore, the metric on the right hand side can also be treated as a function of $g^{mn}(y_a(x))$:

$$g^{mn}(x)=\frac{\partial y^m_a}{\partial x^k}\frac{\partial y^n_a}{\partial x^l}\;g^{kl}(x(y_a)),\qquad(278)$$

with the understanding that $g^{mn}(x)=g^{mn}(y_a(x))$. Components of $g^{mn}(x)$ can be expressed, similar to Eqs. (151)–(153), using the two harmonic potentials:

$$g^{00}(x)\;=\;1+\frac{2}{c^2}w(x)+\frac{2}{c^4}w^2(x)+O(c^{-6}),\qquad(279)$$

$$g^{0\alpha}(x)\;=\;\frac{4}{c^3}w^\alpha(x)+O(c^{-5}),\qquad(280)$$

$$g^{\alpha\beta}(x)\;=\;\gamma^{\alpha\beta}-\gamma^{\alpha\beta}\frac{2}{c^2}w(x)+O(c^{-4}),\qquad(281)$$

where the total (gravitation plus inertia) local scalar $w(x)$ and vector $w^\alpha(x)$ potentials have the following form:

$$w(x)\;=\;\sum_b w_b(x)+\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}+\frac{1}{c^2}\Big\{\frac{\partial\hat{\mathcal{L}}_a}{\partial x^0}+\frac{v^\epsilon_{a_0}}{c}\frac{\partial\hat{\mathcal{L}}_a}{\partial x^\epsilon}+\tfrac{1}{2}\Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}\Big)^2-\Big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)^2\Big\}+\mathcal{O}(c^{-4}),\qquad(282)$$

$$w^\alpha(x)\;=\;\sum_b w^\alpha_b(x)+\tfrac{1}{4}\Big\{\gamma^{\alpha\epsilon}\frac{1}{c}\frac{\partial\hat{\mathcal{L}}_a}{\partial x^\epsilon}+c\frac{\partial\hat{\mathcal{Q}}^\alpha_a}{\partial x^0}+v^\epsilon_{a_0}\frac{\partial\hat{\mathcal{Q}}^\alpha_a}{\partial x^\epsilon}-v^\alpha_{a_0}\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}\Big\}+\mathcal{O}(c^{-2}),\qquad(283)$$

with gravitational potentials in the local frame $w_b(x)$ and $w^\alpha_b(x)$ expressed in global coordinates related to their counterparts in the global frame $w_b(x(y_a))$ and $w^\alpha_b(x(y_a))$ as below

$$w_b(x)\;=\;\Big(1-\frac{2}{c^2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)w_b(x(y_a))+\frac{4}{c^2}v_{a_0\epsilon}w^\epsilon_b(x(y_a))+\mathcal{O}(c^{-4}),\qquad(284)$$

$$w^\alpha_b(x)\;=\;w^\alpha_b(x(y_a))-v^\alpha_{a_0}w_b(x(y_a))+\mathcal{O}(c^{-2}).\qquad(285)$$

Clearly, the potentials $w_b(x)$ and $w^\alpha_b(x)$ introduced by Eqs. (282)–(283) satisfy the following harmonic equations in the local coordinates $\{y^m_a\}$:

$$\Box_{y_a}w(x)=4\pi G\sum_b\Big\{\Big(1-\frac{2}{c^2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)\sigma_b(x(y_a))+\frac{4}{c^2}v_{a_0\epsilon}\sigma^\epsilon_b(x(y_a))+\frac{2}{c^2}\big(\frac{\partial\hat{\mathcal{K}}_a}{\partial x^0}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\big)\sigma_b(x(y_a))+\mathcal{O}(c^{-4})\Big\},\qquad(286)$$

$$\Delta_{y_a}w^\alpha(x)=4\pi G\sum_b\Big\{\sigma^\alpha_b(x(y_a))-v^\alpha_{a_0}\sigma_b(x(y_a))+\mathcal{O}(c^{-2})\Big\}.\qquad(287)$$

With the help of Eqs. (86)–(88) one can verify that both Eqs. (286) and (287) are identically satisfied.

Finally, using the relations $c\partial/\partial y^0_a = c\partial/\partial x^0 + v^\epsilon_{a_0}\partial/\partial x^\epsilon + \mathcal{O}(c^{-2})$ and $\partial/\partial y^\alpha_a = \partial/\partial x^\alpha + \mathcal{O}(c^{-2})$ in Eq. (81), one can verify that potentials $w_b(x)$ and $w^\alpha_b(x)$ satisfy the following continuity equation:

$$\Big(c\frac{\partial}{\partial x^0} + v^\epsilon_{a_0}\frac{\partial}{\partial x^\epsilon}\Big)w(x) + \frac{\partial}{\partial x^\epsilon}w^\epsilon(x) = \mathcal{O}(c^{-2}). \tag{288}$$

These expressions allows us to present the scalar and vector potentials Eqs. (328)–(329), expressed entirely as function of the global coordinates. Remembering Eq. (147) and also the fact that

$$\Box_x w_b(x(y_a)) \;=\; 4\pi G\sigma_b(x(y_a)) + \mathcal{O}(c^{-4}), \qquad \Delta_x w^\alpha_b(x(y_a)) = 4\pi G\sigma^\alpha_b(x(y_a)) + \mathcal{O}(c^{-2}), \tag{289}$$

where solutions are given as:

$$w_b(x(y_a)) \;=\; G\int d^3x'\frac{\sigma_b(y_b(x'))}{|\mathbf{x}-\mathbf{x}'|} + \frac{1}{2c^2}G\frac{c^2\partial^2}{\partial x^{0^2}}\int d^3x'\sigma_b(y_b(x'))|\mathbf{x}-\mathbf{x}'| + \mathcal{O}(c^{-4}), \tag{290}$$

$$w^\alpha_b(x(y_a)) \;=\; G\int d^3x'\frac{\sigma^\alpha_b(y_b(x'))}{|\mathbf{x}-\mathbf{x}'|} + \mathcal{O}(c^{-2}), \tag{291}$$

we can use these results to present the gravitational potentials (290)–(291) in coordinates of the global frame in the form of the integrals over the body's volume as below (where x^m is understood as $x^m = x^m(y_a)$):

$$\begin{aligned} w_b(x) \;=\; & \big(1-\frac{2}{c^2}v_{a_0\epsilon}v^\epsilon_{a_0}\big)G\int d^3x'\frac{\sigma_b(y_b(x'))}{|\mathbf{x}-\mathbf{x}'|} + \frac{4}{c^2}v_{a_0\epsilon}G\int d^3x'\frac{\sigma^\epsilon_b(y_b(x'))}{|\mathbf{x}-\mathbf{x}'|} + \\ & + \frac{1}{2c^2}G\frac{c^2\partial^2}{\partial x^{0^2}}\int d^3x'\sigma_b(y_b(x'))|\mathbf{x}-\mathbf{x}'| + \mathcal{O}(c^{-3}), \end{aligned} \tag{292}$$

$$w^\alpha_b(x) \;=\; G\int d^3x'\frac{\sigma^\alpha_b(y_b(x'))}{|\mathbf{x}-\mathbf{x}'|} - v^\alpha_{a_0}G\int d^3x'\frac{\sigma_b(y_b(x'))}{|\mathbf{x}-\mathbf{x}'|} + \mathcal{O}(c^{-2}). \tag{293}$$

Although the expressions for the scalar inertial potentials have different functional dependence on the transformation functions (i.e., $(\mathcal{K},\mathcal{L},\mathcal{Q}^\alpha)$ vs. $(\hat{\mathcal{K}},\hat{\mathcal{L}},\hat{\mathcal{Q}}^\alpha)$), it is clear that both expressions for $w(y_a)$ given by Eqs. (154) and (282) are identical. The same is true for the inertial vector potentials $w^\alpha(y_a)$ given by (155) and (283).

1. *Determining the structure of* $\hat{\mathcal{K}}_a$

The general solution to Eq. (86) can be thought in the following form:

$$\hat{\mathcal{K}}_a(x) = \hat{\kappa}_{a_0} + \hat{\kappa}_{a_0\mu}r^\mu_a + \delta\hat{\kappa}_a(x) + \mathcal{O}(c^{-4}), \qquad \text{where} \qquad \delta\hat{\kappa}_a(x) = \sum_{k=2}\frac{1}{k!}\hat{\kappa}_{a_0\mu_1\ldots\mu_k}(r^0)r^{\mu_1}_a\cdots r^{\mu_k}_a + \mathcal{O}(c^{-4}), \tag{294}$$

where r^α_a is defined as below:

$$r^\alpha_a = x^\alpha - x^\alpha_{a_0}, \qquad x^\alpha_{a_0} = z^\alpha_{a_0} + c^{-2}\hat{q}^\alpha_{a_0} + \mathcal{O}(c^{-4}), \tag{295}$$

where $x^\alpha_{a_0}(x^0)$ is the position vector of the body a, complete to $\mathcal{O}(c^{-4})$, and expressed as a function of the global time-like coordinate x^0, given by Eq. (47) and $\hat{\kappa}_{a_0\mu_1\ldots\mu_k}(x^0)$ being Cartesian spatial trace-free (STF) tensors [24] which depend only on the time taken at the origin of the coordinate system on the observer's world-line. Substituting this form of the function $\hat{\mathcal{K}}_a$ in the equation (91), we find solution for $\hat{\kappa}_{a_0\mu}$ and $\delta\hat{\kappa}_a$:

$$\hat{\kappa}_{a_0\mu} = cv_{a_0\mu} + \mathcal{O}(c^{-4}), \qquad \delta\hat{\kappa}_a = \mathcal{O}(c^{-4}). \tag{296}$$

As a result, the function $\hat{\mathcal{K}}_a$ that satisfies the harmonic gauge conditions was determined to be

$$\hat{\mathcal{K}}_a(x) = \hat{\kappa}_{a_0} + c(v_{a_0\mu}r^\mu_a) + \mathcal{O}(c^{-4}). \tag{297}$$

This expression completely fixes the spatial dependence of the function $\hat{\mathcal{K}}_a$, but still has unknown dependence on time via function $\hat{\kappa}_{a_0}(x^0)$. This dependence determines the proper time on the observers world-line which will be done later.

2. Determining the structure of $\hat{\mathcal{Q}}^\alpha_a$

The general solution for the function $\hat{\mathcal{Q}}^\alpha_a$ that satisfies Eq. (88) may be presented as a sum of a solution of the inhomogeneous Poisson equation and a solution of the homogeneous Laplace equation. Furthermore, the part of that solution with regular behavior in the vicinity of the world-line may be given in the following form:

$$\hat{\mathcal{Q}}^\alpha_a(x) = -\hat{q}^\alpha_{a_0} + \hat{q}^\alpha_{a_0\mu} r^\mu_a + \tfrac{1}{2}\hat{q}^\alpha_{a_0\mu\nu} r^\mu_a r^\nu_a + \delta\hat{\xi}^\alpha_a(x) + \mathcal{O}(c^{-2}), \tag{298}$$

where $\hat{q}^\alpha_{a_0\mu\nu}$ can be determined directly from Eq. (85) and function $\delta\hat{\xi}^\alpha_a$ satisfies the Laplace equation

$$\gamma^{\epsilon\lambda}\frac{\partial^2}{\partial x^\epsilon \partial x^\lambda}\delta\hat{\xi}^\alpha_a \;=\; \mathcal{O}(c^{-2}). \tag{299}$$

We can see that Eq. (88) can be used to determine $\hat{q}^\alpha_{a_0\mu\nu}$, but would leave the other terms in the equation unspecified. To determine these terms, we use Eq. (92) (which is equivalent to Eq. (88)) together with Eq. (91), and obtain:

$$v^\alpha_{a_0} v^\beta_{a_0} + \gamma^{\alpha\lambda}\frac{\partial \hat{\mathcal{Q}}^\beta_a}{\partial x^\lambda} + \gamma^{\beta\lambda}\frac{\partial \hat{\mathcal{Q}}^\alpha_a}{\partial x^\lambda} + 2\gamma^{\alpha\beta}\Big(\frac{\partial \hat{\mathcal{K}}_a}{\partial x^0} + \tfrac{1}{2} v_{a_0\epsilon} v^\epsilon_{a_0}\Big) \;=\; \mathcal{O}(c^{-2}). \tag{300}$$

Using the intermediate solution for the function $\hat{\mathcal{K}}_a$ from Eq. (297) in Eq. (300), we obtain the following equation for $\hat{\mathcal{Q}}^\alpha_a$:

$$v^\alpha_{a_0} v^\beta_{a_0} + \gamma^{\alpha\lambda}\frac{\partial \hat{\mathcal{Q}}^\beta_a}{\partial x^\lambda} + \gamma^{\beta\lambda}\frac{\partial \hat{\mathcal{Q}}^\alpha_a}{\partial x^\lambda} + 2\gamma^{\alpha\beta}\Big(\frac{\partial \hat{\kappa}_{a_0}}{\partial x^0} - \tfrac{1}{2} v_{a_0\epsilon} v^\epsilon_{a_0} + a_{0\epsilon} r^\epsilon_a\Big) \;=\; \mathcal{O}(c^{-2}). \tag{301}$$

A trial solution to Eq. (301) may be given in the following general from:

$$\begin{aligned}\hat{\mathcal{Q}}^\alpha_a(x) \;=\; & -\hat{q}^\alpha_{a_0} + c_1 v^\alpha_{a_0} v_{a_0\epsilon} r^\epsilon_a + c_2 v_{a_0\epsilon} v^\epsilon_{a_0} r^\alpha_a + c_3 a^\alpha_{a_0} r_{a\epsilon} r^\epsilon_a + c_4 a_{a_0\epsilon} r^\epsilon_a r^\alpha_a + c_5\big(\frac{\partial \hat{\kappa}_{a_0}}{\partial x^0} - \tfrac{1}{2} v_{a_0\epsilon} v^\epsilon_{a_0}\big) r^\alpha_a - \\ & - \hat{\omega}^{\alpha\epsilon}_{a_0} r_{a\epsilon} + \delta\hat{\xi}^\mu_a(x) + \mathcal{O}(c^{-2}),\end{aligned} \tag{302}$$

where $\hat{q}^\alpha_{a_0}$ and antisymmetric matrix $\hat{\omega}^{\alpha\epsilon}_{a_0} = -\hat{\omega}^{\epsilon\alpha}_{a_0}$ are some functions of time, and $c_1, ..., c_5$ are constants and $\delta\hat{\xi}^\mu_a(x)$ is given by Eq. (299) and is at least of the third order in spatial coordinates r^μ_a, namely $\delta\hat{\xi}^\mu_a(x) \propto \mathcal{O}(|r^\mu_a|^3)$. Direct substitution of Eq. (302) into Eq. (301) results in the following unique solution for these coefficients:

$$c_1 = -\frac{1}{2}, \quad c_2 = 0, \quad c_3 = \frac{1}{2}, \quad c_4 = -1, \quad c_5 = -1. \tag{303}$$

As a result, function $\hat{\mathcal{Q}}^\alpha_a$ has the following structure

$$\hat{\mathcal{Q}}^\alpha_a(x) \;=\; -\hat{q}^\alpha_{a_0} - \Big(\tfrac{1}{2} v^\alpha_{a_0} v^\epsilon_{a_0} + \hat{\omega}^{\alpha\epsilon}_{a_0} + \gamma^{\alpha\epsilon}\big(\frac{\partial \hat{\kappa}_{a_0}}{\partial x^0} - \tfrac{1}{2} v_{a_0\lambda} v^\lambda_{a_0}\big)\Big) r_{a\epsilon} - a_{a_0\epsilon}\Big(r^\alpha_a r^\epsilon_a - \tfrac{1}{2}\gamma^{\alpha\epsilon} r_{a\lambda} r^\lambda_a\Big) + \delta\hat{\xi}^\alpha_a(x) + \mathcal{O}(c^{-2}), \tag{304}$$

where $\hat{q}^\alpha_{a_0}$ and $\hat{\omega}^{\alpha\epsilon}_{a_0}$ are yet to be determined.

By substituting Eq. (304) into Eq. (301), we see that the function $\delta\hat{\xi}^\alpha_a(x)$ in Eq. (304) must satisfy the equation:

$$\frac{\partial}{\partial x_\alpha}\delta\hat{\xi}^\beta_a + \frac{\partial}{\partial x_\beta}\delta\hat{\xi}^\alpha_a = \mathcal{O}(c^{-2}). \tag{305}$$

We keep in mind that the function $\delta\hat{\xi}^\alpha_a(x)$ must also satisfy Eq. (299). The solution to the partial differential equation (299) with regular behavior on the world-line (i.e., when $|\mathbf{r}_a| \to 0$) can be given in powers of r^μ_a as

$$\delta\hat{\xi}^\alpha_a(x) = \sum_{k\geq 3}\frac{1}{k!}\delta\hat{\xi}^\alpha_{a_0\,\mu_1...\mu_k}(x^0) r^{\mu_1}_a \cdots r^{\mu_k}_a + \mathcal{O}(|r^\mu_a|^K) + \mathcal{O}(c^{-2}), \tag{306}$$

where $\delta\hat{\xi}^\alpha_{a_0\,\mu_1...\mu_k}(x^0)$ being STF tensors that depend only on time-like coordinate x_0. Using the solution (306) in Eq. (305), we can see that $\delta\hat{\xi}^\alpha_{a_0\,\mu_1...\mu_n}$ is also antisymmetric with respect to the index α and any of the spatial indices $\mu_1...\mu_k$. Combination of these two conditions suggests that $\delta\hat{\xi}^\alpha_{a_0\,\mu_1...\mu_n} = 0$ for all $k \geq 3$, thus

$$\delta\hat{\xi}^\alpha_a(x) = \mathcal{O}(c^{-2}). \tag{307}$$

Therefore, application of the harmonic gauge conditions leads to the following structure of function $\hat{\mathcal{Q}}^\alpha_a$:

$$\hat{\mathcal{Q}}^\alpha_a(x) \;=\; -\hat{q}^\alpha_{a_0} - \Big(\tfrac{1}{2} v^\alpha_{a_0} v^\epsilon_{a_0} + \hat{\omega}^{\alpha\epsilon}_{a_0} + \gamma^{\alpha\epsilon}\big(\frac{\partial \hat{\kappa}_{a_0}}{\partial x^0} - \tfrac{1}{2} v_{a_0\lambda} v^\lambda_{a_0}\big)\Big) r_{a\epsilon} - a_{a_0\epsilon}\Big(r^\alpha_a r^\epsilon_a - \tfrac{1}{2}\gamma^{\alpha\epsilon} r_{a\lambda} r^\lambda_a\Big) + \mathcal{O}(c^{-2}), \tag{308}$$

where $\hat{q}^\alpha_{a_0}, \hat{\omega}^{\alpha\epsilon}_{a_0}$ and $\hat{\kappa}_{a_0}$ are yet to be determined.

3. Determining the structure of $\hat{\mathcal{L}}_a$

We now turn our attention to the second gauge condition on the temporal coordinate transformation, Eq. (87). Using the intermediate solution (297) for the function $\hat{\mathcal{K}}_a$, we obtain the following equation for $\hat{\mathcal{L}}_a$:

$$\gamma^{\epsilon\lambda}\frac{\partial^2\hat{\mathcal{L}}_a}{\partial x^\epsilon\partial x^\lambda} = -c^2\frac{\partial^2\hat{\mathcal{K}}_a}{\partial {x^0}^2}+\mathcal{O}(c^{-2}) = c\big(2v_{a_0\epsilon}a^\epsilon_{a_0}-\dot{a}_{a_0\epsilon}r^\epsilon_a\big)-c^2\frac{\partial}{\partial x^0}\Big(\frac{\partial\hat{\kappa}_{a_0}}{\partial x^0}-\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)+\mathcal{O}(c^{-2}). \tag{309}$$

The general solution of Eq. (309) for $\hat{\mathcal{L}}_a$ may be presented as a sum of a solution $\delta\hat{\mathcal{L}}_a$ for the inhomogeneous Poisson equation and a solution $\delta\hat{\mathcal{L}}_{a0}$ of the homogeneous Laplace equation. A trial solution of the inhomogeneous equation to this equation, $\delta\hat{\mathcal{L}}_a$, is sought in the following form:

$$\delta\hat{\mathcal{L}}_a(x) = ck_1(v_{a_0\epsilon}a^\epsilon_{a_0})(r_{a\mu}r^\mu_a)+ck_2(\dot{a}_{a_0\epsilon}r^\epsilon_a)(r_{a\nu}r^\nu_a)-k_3c^2\frac{\partial}{\partial x^0}\Big(\frac{\partial\kappa_{a_0}}{\partial x^0}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)(r_{a\nu}r^\nu_a)+\mathcal{O}(c^{-2}), \tag{310}$$

where k_1, k_2, k_3 are some constants. Direct substitution of Eq. (310) into Eq. (309) yields the following values for these coefficients:

$$k_1=\frac{1}{3},\quad k_2=\frac{1}{10},\quad k_3=\frac{1}{6}. \tag{311}$$

As a result, the solution for $\delta\hat{\mathcal{L}}_a$ that satisfies the harmonic gauge conditions has the following form:

$$\delta\hat{\mathcal{L}}_a(x) = \tfrac{1}{3}c\big(v_{a_0\epsilon}a^\epsilon_{a_0}\big)(r_{a\nu}r^\nu_a)-\tfrac{1}{10}c(\dot{a}_{a_0\epsilon}r^\epsilon_a)(r_{a\nu}r^\nu_a)-\tfrac{1}{6}c^2\frac{\partial}{\partial x^0}\Big(\frac{\partial\kappa_{a_0}}{\partial x^0}+\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)(r_{a\nu}r^\nu_a)+\mathcal{O}(c^{-2}). \tag{312}$$

The solution for the homogeneous equation with regular behavior on the world-line (i.e., when $|r^\mu_a|\to 0$) may be presented as follows:

$$\hat{\mathcal{L}}_{a0}(x) = \hat{\ell}_{a_0}(x^0)+\hat{\ell}_{a_0\lambda}(x^0)\,r^\lambda_a+\tfrac{1}{2}\hat{\ell}_{a_0\lambda\mu}(x^0)\,r^\lambda_a r^\mu_a+\delta\hat{\ell}_a(x)+\mathcal{O}(c^{-2}), \tag{313}$$

where $\hat{\ell}_{a_0\lambda\mu}$ is the STF tensor of second rank and $\delta\hat{\ell}_a$ is a function formed from the similar STF tensors of higher order

$$\delta\hat{\ell}_a(x)=\sum_{k\geq 3}\frac{1}{k!}\delta\hat{\ell}_{a_0\,\mu_1\ldots\mu_k}(x^0)r^{\mu_1}_a\cdots r^{\mu_k}_a+\mathcal{O}(|r^\mu_a|^K)+\mathcal{O}(c^{-2}). \tag{314}$$

Finally, the general solution of the Eq. (309) may be presented as a sum of the special solution of inhomogeneous equation $\delta\hat{\mathcal{L}}_a$ and solution, $\hat{\mathcal{L}}_{a0}$, to the homogeneous equation, $\Delta\hat{\mathcal{L}}_a=0$. Therefore, the general solution for the gauge equations for function $\hat{\mathcal{L}}_a(y)=\hat{\mathcal{L}}_{a0}+\delta\hat{\mathcal{L}}_a$ has the following form:

$$\begin{aligned}\hat{\mathcal{L}}_a(x) &= \hat{\ell}_{a_0}+\hat{\ell}_{a_0\lambda}\,r^\lambda_a+\tfrac{1}{2}\hat{\ell}_{a_0\lambda\mu}\,r^\lambda_a r^\mu_a+\tfrac{1}{3}c\big(v_{a_0\epsilon}a^\epsilon_{a_0}\big)(r_{a\nu}r^\nu_a)-\tfrac{1}{10}c(\dot{a}_{a_0\epsilon}r^\epsilon_a)(r_{a\nu}r^\nu_a)-\\ &-\tfrac{1}{6}c^2\frac{\partial}{\partial x^0}\Big(\frac{\partial\hat{\kappa}_{a_0}}{\partial x^0}-\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)(r_{a\nu}r^\nu_a)+\delta\hat{\ell}_a(x)+\mathcal{O}(c^{-2}).\end{aligned} \tag{315}$$

We successfully determined the structure of the transformation functions $\mathcal{K}_a, \mathcal{Q}^\alpha_a$ and $\mathcal{L}_a$, which are imposed by the harmonic gauge conditions. Specifically, the harmonic structure for $\mathcal{K}$ is given by Eq. (296), the function $\mathcal{Q}^\alpha_a$ was determined to have the structure given by Eq. (309), and the structure for $\mathcal{L}_a$ is given by Eq. (315). The structure of the functions $\hat{\mathcal{K}}_a, \hat{\mathcal{Q}}^\alpha_a$ and $\hat{\mathcal{L}}_a$ given in the following form:

$$\hat{\mathcal{K}}_a(x) = \hat{\kappa}_{a_0}+c(v_{a_0\mu}r^\mu_a)+\mathcal{O}(c^{-4}), \tag{316}$$

$$\hat{\mathcal{Q}}^\alpha_a(x) = -\hat{q}^\alpha_{a_0}-\Big(\tfrac{1}{2}v^\alpha_{a_0}v^\epsilon_{a_0}+\hat{\omega}^{\alpha\epsilon}_{a_0}+\gamma^{\alpha\epsilon}\big(\frac{\partial\hat{\kappa}_{a_0}}{\partial x^0}-\tfrac{1}{2}v_{a_0\lambda}v^\lambda_{a_0}\big)\Big)r_{a\epsilon}-a_{a_0\epsilon}\Big(r^\alpha_a r^\epsilon_a-\tfrac{1}{2}\gamma^{\alpha\epsilon}r_{a\lambda}r^\lambda_a\Big)+\mathcal{O}(c^{-2}), \tag{317}$$

$$\begin{aligned}\hat{\mathcal{L}}_a(x) &= \hat{\ell}_{a_0}+\hat{\ell}_{a_0\lambda}\,r^\lambda_a+\tfrac{1}{2}\hat{\ell}_{a_0\lambda\mu}\,r^\lambda_a r^\mu_a+\tfrac{1}{6}c\Big(2\big(v_{a_0\epsilon}a^\epsilon_{a_0}\big)-c\frac{\partial}{\partial x^0}\big(\frac{\partial\hat{\kappa}_{a_0}}{\partial x^0}-\tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\big)\Big)(r_{a\lambda}r^\lambda_a)-\\ &-\tfrac{1}{10}c(\dot{a}_{a_0\epsilon}r^\epsilon_a)(r_{a\nu}r^\nu_a)+\delta\hat{\ell}_a(x)+\mathcal{O}(c^{-2}).\end{aligned} \tag{318}$$

Note that the harmonic gauge conditions allow us to reconstruct the structure of the functions with respect to spatial coordinate x^μ. The time-dependent functions $\hat{\kappa}_{a_0}, \hat{q}^\alpha_{a_0}, \hat{\omega}^{\alpha\epsilon}_{a_0}, \hat{\ell}_{a_0}, \hat{\ell}_{a_0\lambda}, \hat{\ell}_{a_0\lambda\mu}$, and $\delta\hat{\ell}_{a_0\,\mu_1\ldots\mu_k}$ cannot be determined from the gauge conditions alone. Instead, just as we did in Sec. IV, we resort to a set of dynamic conditions to define unambiguously the proper reference frame of a moving observer.

B. Finding the form of the coordinate transformation functions

In the case of the metric tensor given by the expressions Eqs. (279)–(281), the conditions Eqs. (208)–(209) with $r^\epsilon_a = x^\epsilon - x^\epsilon_{a_0}$ given by Eq. (295) lead to the following set of equations.

$$\lim_{|\mathbf{r}_a|\to 0} w(x) \;=\; w_a(x) + \mathcal{O}(c^{-4}), \qquad \lim_{|\mathbf{r}_a|\to 0} \frac{\partial w(x)}{\partial x^\beta} = \frac{\partial w_a(x)}{\partial x^\beta} + \mathcal{O}(c^{-4}), \tag{319}$$

$$\lim_{|\mathbf{r}_a|\to 0} w^\alpha(x) \;=\; w^\alpha_a(x) + \mathcal{O}(c^{-2}), \qquad \lim_{|\mathbf{r}_a|\to 0} \frac{\partial w^\alpha(x)}{\partial x^\beta} = \frac{\partial w^\alpha_a(x)}{\partial x^\beta} + \mathcal{O}(c^{-2}). \tag{320}$$

Imposing the conditions given by Eqs. (319)–(320) on the potentials w and w^α, which are in turn given by Eqs. (282)–(283), results in the following set of partial differential equations set on the world-line of the local observer expressed via global coordinates $\{x^m\}$:

$$\sum_{b\neq a} \bar{w}_b + \frac{\partial \hat{\kappa}_{a_0}}{\partial x^0} - \tfrac{1}{2} v_{a_0\epsilon} v^\epsilon_{a_0} - c^{-2} v_{a_0\epsilon}\dot{\hat{q}}^\epsilon_{a_0} + $$

$$+\frac{1}{c^2}\Big\{\frac{\partial \hat{\mathcal{L}}_a}{\partial x^0} + \frac{v^\epsilon_{a_0}}{c}\frac{\partial \hat{\mathcal{L}}_a}{\partial x^\epsilon} + \tfrac{1}{2}\Big(\frac{\partial \hat{\kappa}_{a_0}}{\partial x^0} - v_{a_0\epsilon}v^\epsilon_{a_0}\Big)^2 - \Big(\frac{\partial \hat{\kappa}_{a_0}}{\partial x^0} - \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)^2\Big\} \;=\; \mathcal{O}(c^{-4}), \tag{321}$$

$$\sum_{b\neq a} \frac{\partial \bar{w}_b}{\partial x^\beta} + a_{a_0\beta} + \frac{1}{c^2}\Big\{\frac{\partial^2 \hat{\mathcal{L}}_a}{\partial x^\beta \partial x^0} + \frac{v^\epsilon_{a_0}}{c}\frac{\partial^2 \hat{\mathcal{L}}_a}{\partial x^\beta \partial x^\epsilon} - a_{a_0\beta}\frac{\partial \hat{\kappa}_{a_0}}{\partial x^0}\Big\} \;=\; \mathcal{O}(c^{-4}), \tag{322}$$

$$\sum_{b\neq a} \bar{w}^\alpha_b + \tfrac{1}{4}\Big\{\gamma^{\alpha\epsilon}\frac{1}{c}\frac{\partial \hat{\mathcal{L}}_a}{\partial x^\epsilon} + c\frac{\partial \hat{\mathcal{Q}}^\alpha_a}{\partial x^0} + v^\epsilon_{a_0}\frac{\partial \hat{\mathcal{Q}}^\alpha_a}{\partial x^\epsilon} - v^\alpha_{a_0}\big(\frac{\partial \hat{\kappa}_{a_0}}{\partial x^0} - v_{a_0\epsilon}v^\epsilon_{a_0}\big)\Big\} \;=\; \mathcal{O}(c^{-2}), \tag{323}$$

$$\sum_{b\neq a} \frac{\partial \bar{w}^b_\alpha}{\partial x^\beta} + \tfrac{1}{4}\Big\{\frac{1}{c}\frac{\partial^2 \hat{\mathcal{L}}_a}{\partial x^\alpha \partial x^\beta} + c\gamma_{\alpha\lambda}\frac{\partial^2 \hat{\mathcal{Q}}^\lambda_a}{\partial x^0 \partial x^\beta} + \gamma_{\alpha\lambda}v^\epsilon_{a_0}\frac{\partial^2 \hat{\mathcal{Q}}^\lambda_a}{\partial x^\epsilon \partial x^\beta} - v_{a_0\alpha}a_{a_0\beta}\Big\} \;=\; \mathcal{O}(c^{-2}), \tag{324}$$

with all the quantifies above now being the functions of the global time-like coordinate x^0.

The equations above can now be used to determine uniquely the form of the coordinate transformation functions. From the first two conditions (321) and (322) above, we immediately obtain

$$\sum_{b\neq a} \bar{U}_b + \frac{\partial \hat{\kappa}_{a_0}}{\partial x^0} - \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0} - c^{-2}v_{a_0\epsilon}\dot{\hat{q}}^\epsilon_{a_0} \;=\; \mathcal{O}(c^{-4}), \tag{325}$$

$$\sum_{b\neq a} \frac{\partial \bar{U}_b}{\partial x^\beta} + a_{a_0\beta} \;=\; \mathcal{O}(c^{-4}), \tag{326}$$

where we split the scalar potential $w_b(x)$ defined in Eq. (284) into Newtonian (U_b) and post-Newtonian (δw_b) parts:

$$w_b = U_b + \frac{1}{c^2}\delta w_b + \mathcal{O}(c^{-4}), \tag{327}$$

so, that gravitational scalar and vector potentials defined by Eqs. (284)–(285) take the following form:

$$w_b(y_a(x)) \;=\; U_b(x) + \frac{1}{c^2}\Big(\delta w_b(x) - 2(v_{a_0\epsilon}v^\epsilon_{a_0})U_b(x) + 4v_{a_0\epsilon}w^\epsilon_b(x)\Big) + \mathcal{O}(c^{-4}), \tag{328}$$

$$w^\alpha_b(y_a(x)) \;=\; w^\alpha_b(x) - v^\alpha_{a_0}U_b(x) + \mathcal{O}(c^{-2}). \tag{329}$$

Eqs. (325) and (297) allow us to determine function $\hat{\kappa}_{a_0}$ and to present the solution for $\hat{\mathcal{K}}_a$ as below:

$$\hat{\mathcal{K}}_a(x) = -\int_{x^0_0}^{x^0}\Big(\sum_{b\neq a}\bar{U}_b - \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0} - c^{-2}v_{a_0\epsilon}\dot{\hat{q}}^\epsilon_{a_0}\Big)dx'^0 + c(v_{a_0\epsilon}r^\epsilon_a) + \mathcal{O}(c^{-2}). \tag{330}$$

Substituting this expression into Eq. (317), we can determine $\hat{\mathcal{Q}}^\alpha_a$:

$$\hat{\mathcal{Q}}^\alpha_a(x) \;=\; -\hat{q}^\alpha_{a_0} + \Big(\gamma^{\alpha\epsilon}\sum_{b\neq a}\bar{U}_b - \tfrac{1}{2}v^\alpha_{a_0}v^\epsilon_{a_0} - \hat{\omega}^{\alpha\epsilon}_{a_0}\Big)r_{a\epsilon} - a_{a_0\epsilon}\Big(r^\alpha_a r^\epsilon_a - \tfrac{1}{2}\gamma^{\alpha\epsilon}r_{a\lambda}r^\lambda_a\Big) + \mathcal{O}(c^{-2}). \tag{331}$$

Finally, the general solution, given by Eq. (315), for the function $\hat{\mathcal{L}}_a$ takes the following form:

$$\hat{\mathcal{L}}_a(x) \;=\; \hat{\ell}_{a_0} + \hat{\ell}_{a_0\lambda}\, r_a^\lambda + \tfrac{1}{2}\Big(\hat{\ell}_{a_0\lambda\mu} + c\gamma_{\lambda\mu}\Big(\tfrac{2}{3}\big(v_{a_0\epsilon}a^\epsilon_{a_0}\big) + \tfrac{1}{3}c\frac{\partial}{\partial x^0}\sum_{b\neq a}\bar{U}_b\Big)r_a^\lambda r_a^\mu - \tfrac{1}{10}c(\dot{a}_{a_0\epsilon}r_a^\epsilon)(r_{a\nu}r_a^\nu) + \\ +\; \delta\hat{\ell}_a(x) + \mathcal{O}(c^{-2}). \tag{332}$$

This equation allows us to proceed onto finding the function $\hat{\mathcal{L}}_a$. To do this, we rely on the yet-unused parts of Eqs. (321)–(324) which take the following form

$$\sum_{b\neq a}\delta\bar{w}_b + \Big\{\frac{\partial\hat{\mathcal{L}}_a}{\partial x^0} + \frac{v^\epsilon_{a_0}}{c}\frac{\partial\hat{\mathcal{L}}_a}{\partial x^\epsilon} + \tfrac{1}{2}\Big(\frac{\partial\hat{\kappa}_{a_0}}{\partial x^0} - v_{a_0\epsilon}v^\epsilon_{a_0}\Big)^2 - \Big(\frac{\partial\hat{\kappa}_{a_0}}{\partial x^0} - \tfrac{1}{2}v_{a_0\epsilon}v^\epsilon_{a_0}\Big)^2\Big\} \;=\; \mathcal{O}(c^{-2}), \tag{333}$$

$$\sum_{b\neq a}\frac{\partial\delta\bar{w}_b}{\partial x^\beta} + \Big\{\frac{\partial^2\hat{\mathcal{L}}_a}{\partial x^\beta\partial x^0} + \frac{v^\epsilon_{a_0}}{c}\frac{\partial^2\hat{\mathcal{L}}_a}{\partial x^\beta\partial x^\epsilon} - a_{a_0\beta}\frac{\partial\hat{\kappa}_{a_0}}{\partial x^0}\Big\} \;=\; \mathcal{O}(c^{-2}), \tag{334}$$

$$\sum_{b\neq a}\bar{w}^\alpha_b + \tfrac{1}{4}\Big\{\gamma^{\alpha\epsilon}\frac{1}{c}\frac{\partial\hat{\mathcal{L}}_a}{\partial x^\epsilon} + c\frac{\partial\hat{\mathcal{Q}}^\alpha_a}{\partial x^0} + v^\epsilon_{a_0}\frac{\partial\hat{\mathcal{Q}}^\alpha_a}{\partial x^\epsilon} - v^\alpha_{a_0}\big(\frac{\partial\hat{\kappa}_{a_0}}{\partial x^0} - v_{a_0\epsilon}v^\epsilon_{a_0}\big)\Big\} \;=\; \mathcal{O}(c^{-2}), \tag{335}$$

$$\sum_{b\neq a}\frac{\partial\bar{w}^b_\alpha}{\partial x^\beta} + \tfrac{1}{4}\Big\{\frac{1}{c}\frac{\partial^2\hat{\mathcal{L}}_a}{\partial x^\alpha\partial x^\beta} + c\gamma_{\alpha\lambda}\frac{\partial^2\hat{\mathcal{Q}}^\lambda_a}{\partial x^0\partial x^\beta} + \gamma_{\alpha\lambda}v^\epsilon_{a_0}\frac{\partial^2\hat{\mathcal{Q}}^\lambda_a}{\partial x^\epsilon\partial x^\beta} - v_{a_0\alpha}a_{a_0\beta}\Big\} \;=\; \mathcal{O}(c^{-2}). \tag{336}$$

Using the expressions for the functions $\hat{\mathcal{K}}_a$, $\hat{\mathcal{Q}}^\alpha_a$ given by Eqs. (330) and (331) correspondingly together with the expression (332) for $\hat{\mathcal{L}}_a$, and also using transformation rule for the gravitational potentials given by Eqs. (328) and (329) together with the Newtonian equations of motion (326), we can now obtain the equations needed to determine the remaining unknown functions $\hat{\ell}_{a_0}, \hat{\ell}_{a_0\lambda}$, $\hat{\ell}_{a_0\lambda\mu}$, $\hat{\omega}^{\alpha\beta}_{a_0}$ and $\hat{q}^\alpha_{a_0}$.

Eq. (333) leads to the following equation for $\hat{\ell}_{a_0}$:

$$\tfrac{1}{c}\dot{\hat{\ell}}_{a_0} \;=\; -\tfrac{1}{8}(v_{a_0\epsilon}v^\epsilon_{a_0})^2 + \tfrac{3}{2}(v_{a_0\epsilon}v^\epsilon_{a_0})\sum_{b\neq a}\bar{U}_b - 4v_{a_0\epsilon}\sum_{b\neq a}\bar{w}^\epsilon_b + \tfrac{1}{2}(\sum_{b\neq a}\bar{U}_b)^2 - \sum_{b\neq a}\delta\bar{w}_b + \mathcal{O}(c^{-2}). \tag{337}$$

Next, Eq. (334) results in the equation for $\dot{\hat{\ell}}^\alpha_{a_0}$:

$$\tfrac{1}{c}\dot{\hat{\ell}}^\alpha_{a_0} \;=\; -\sum_{b\neq a}\frac{\partial}{\partial x_\alpha}\delta\bar{w}_b - \tfrac{3}{2}(v_{a_0\epsilon}v^\epsilon_{a_0})a^\alpha_{a_0} - a^\alpha_{a_0}\sum_{b\neq a}\bar{U}_b - 4v_{a_0\epsilon}\sum_{b\neq a}\partial^\alpha\bar{w}^\epsilon_b + \mathcal{O}(c^{-2}). \tag{338}$$

From Eq. (335) we determine $\hat{\ell}^\alpha_{a_0}$:

$$\tfrac{1}{c}\hat{\ell}^\alpha_{a_0} \;=\; \dot{\hat{q}}^\alpha_{a_0} - \tfrac{1}{2}v^\alpha_{a_0}(v_{a_0\epsilon}v^\epsilon_{a_0}) + 3v^\alpha_{a_0}\sum_{b\neq a}\bar{U}_b - 4\sum_{b\neq a}\bar{w}^\alpha_b + \mathcal{O}(c^{-2}). \tag{339}$$

Furthermore, Eq. (336) leads us to the following solution for $\hat{\ell}^{\alpha\beta}_{a_0}$:

$$\tfrac{1}{c}\hat{\ell}^{\alpha\beta}_{a_0} \;=\; -4\sum_{b\neq a}\frac{\partial}{\partial x_\beta}\bar{w}^\alpha_b - \tfrac{4}{3}\gamma^{\alpha\beta}\sum_{b\neq a}c\frac{\partial\bar{U}_b}{\partial x^0} - \tfrac{5}{2}v^\alpha_{a_0}a^\beta_{a_0} + \tfrac{1}{2}v^\beta_{a_0}a^\beta_{a_0} - \tfrac{2}{3}\gamma^{\alpha\beta}v_{a_0\epsilon}a^\epsilon_{a_0} + \dot{\hat{\omega}}^{\alpha\beta}_{a_0} + \mathcal{O}(c^{-2}). \tag{340}$$

Using the same argument concerning the symmetry properties of $\hat{\ell}^{\alpha\beta}_{a_0}$ that led to Eq. (232), we find the following unique choice for the anti-symmetric matrix $\dot{\hat{\omega}}^{\alpha\beta}_{a_0}$ that symmetrize Eq. (340):

$$\dot{\hat{\omega}}^{\alpha\beta}_{a_0} \;=\; \tfrac{3}{2}(v^\alpha_{a_0}a^\beta_{a_0} - v^\beta_{a_0}a^\alpha_{a_0}) - 2\sum_{b\neq a}\Big(\frac{\partial\bar{w}^\beta_b}{\partial x_\alpha} - \frac{\partial\bar{w}^\alpha_b}{\partial x_\beta}\Big) + \mathcal{O}(c^{-2}), \tag{341}$$

which, if not accounting for Newtonian equation of motion (326), actually has the following form:

$$\dot{\hat{\omega}}^{\alpha\beta}_{a_0} \;=\; -\tfrac{1}{2}(v^\alpha_{a_0}a^\beta_{a_0} - v^\beta_{a_0}a^\alpha_{a_0}) - 2\sum_{b\neq a}\Big(v^\alpha_{a_0}\frac{\partial\bar{U}_b}{\partial x_\beta} - v^\beta_{a_0}\frac{\partial\bar{U}_b}{\partial x_\alpha}\Big) - 2\sum_{b\neq a}(\partial^\alpha\bar{w}^\beta_b - \partial^\beta\bar{w}^\alpha_b) + \mathcal{O}(c^{-2}). \tag{342}$$

The first term in the equation above is the special relativistic Thomas precession, the second term is the geodetic precession and the last term is the Lense-Thirring precession.

The result (341) allows us to present Eq. (340) for the function $\hat{\ell}^{\alpha\beta}_{a_0}$ in the following form:

$$\tfrac{1}{c}\hat{\ell}^{\alpha\beta}_{a_0} = -2\sum_{b\neq a}\Big(\frac{\partial\bar{w}^{\beta}_b}{\partial x_\alpha}+\frac{\partial\bar{w}^{\alpha}_b}{\partial x_\beta}\Big)-\tfrac{4}{3}\gamma^{\alpha\beta}\sum_{b\neq a}c\frac{\partial\bar{U}_b}{\partial x^0}-\Big(v^{\alpha}_{a_0}a^{\beta}_{a_0}+v^{\beta}_{a_0}a^{\alpha}_{a_0}+\tfrac{2}{3}\gamma^{\alpha\beta}v_{a_0\epsilon}a^{\epsilon}_{a_0}\Big)+\mathcal{O}(c^{-2}). \tag{343}$$

Finally, Eqs. (338) and (339) allow us to determine the equation for $\hat{q}^{\alpha}_{a_0}$. Indeed, differentiating Eq. (339) with respect to time and subtracting the result from Eq. (338), we obtain:

$$\begin{aligned}\ddot{\hat{q}}^{\alpha}_{a_0}(x^0) &= -\gamma^{\alpha\epsilon}\sum_{b\neq a}\frac{\partial\delta\bar{w}_b}{\partial x^\epsilon}+4\sum_{b\neq a}c\frac{\partial\bar{w}^{\alpha}_b}{\partial x^0}-4v_{a_0\lambda}\gamma^{\alpha\epsilon}\sum_{b\neq a}\frac{\partial\bar{w}^{\lambda}_b}{\partial x^\epsilon}-4a^{\alpha}_{a_0}\sum_{b\neq a}\bar{U}_b- \\ &\quad -3v^{\alpha}_{a_0}\sum_{b\neq a}c\frac{\partial\bar{U}_b}{\partial x^0}-(v_{a_0\epsilon}v^{\epsilon}_{a_0})a^{\alpha}_{a_0}+v^{\alpha}_{a_0}(v_{a_0\epsilon}a^{\epsilon}_{a_0})+\mathcal{O}(c^{-2}).\end{aligned} \tag{344}$$

By combining Eqs. (326) and (344), we obtain the equations of motion of the body a with respect to the inertial reference frame $\{x^m\}$:

$$\begin{aligned}\ddot{x}^{\alpha}_{a_0}(x^0) &= \ddot{z}^{\alpha}_{a_0}+c^{-2}\ddot{\hat{q}}^{\alpha}_{a_0}+\mathcal{O}(c^{-4}) = \\ &= -\gamma^{\alpha\epsilon}\sum_{b\neq a}\frac{\partial\bar{w}_b}{\partial x^\epsilon}+\frac{1}{c^2}\Big(4\sum_{b\neq a}c\frac{\partial\bar{w}^{\alpha}_b}{\partial x^0}-4v_{a_0\lambda}\gamma^{\alpha\epsilon}\sum_{b\neq a}\frac{\partial\bar{w}^{\lambda}_b}{\partial x^\epsilon}-4a^{\alpha}_{a_0}\sum_{b\neq a}\bar{U}_b+ \\ &\qquad -3v^{\alpha}_{a_0}\sum_{b\neq a}c\frac{\partial\bar{U}_b}{\partial x^0}-(v_{a_0\epsilon}v^{\epsilon}_{a_0})a^{\alpha}_{a_0}+v^{\alpha}_{a_0}(v_{a_0\epsilon}a^{\epsilon}_{a_0})\Big)+\mathcal{O}(c^{-4}).\end{aligned} \tag{345}$$

The equations of motion (345) establish the correspondence barycentric equations of motion of the body a.

C. Summary of results for the inverse transformation

In this section, we sought to write the transformation between global coordinates $\{x^k\}$ of the barycentric reference system and local coordinates $\{y^k_a\}$ introduced in the proper body-centric coordinate reference system associated with the body a in the form:

$$y^0_a = x^0+c^{-2}\hat{\mathcal{K}}_a(x^k)+c^{-4}\hat{\mathcal{L}}_a(x^k)+\mathcal{O}(c^{-6}), \tag{346}$$

$$y^{\alpha}_a = x^{\alpha}-z^{\alpha}_{a_0}(x^0)+c^{-2}\hat{\mathcal{Q}}^{\alpha}_a(x^k)+\mathcal{O}(c^{-4}). \tag{347}$$

Imposing the same conditions discussed in Sec. IV C, we were able to find explicit forms for the transformation functions $\hat{\mathcal{K}}_a, \hat{\mathcal{L}}_a$ and $\hat{\mathcal{Q}}^{\alpha}_a$:

$$\hat{\mathcal{K}}_a(x) = -\int_{x^0_0}^{x^0}\Big(\sum_{b\neq a}\bar{U}_b-\tfrac{1}{2}v_{a_0\epsilon}v^{\epsilon}_{a_0}-c^{-2}v_{a_0\epsilon}\dot{\hat{q}}^{\epsilon}_{a_0}\Big)dx'^0+c(v_{a_0\epsilon}r^{\epsilon}_a)+\mathcal{O}(c^{-4}), \tag{348}$$

$$\hat{\mathcal{Q}}^{\alpha}_a(x) = -\hat{q}^{\alpha}_{a_0}+\Big(\gamma^{\alpha\epsilon}\sum_{b\neq a}\bar{U}_b-\tfrac{1}{2}v^{\alpha}_{a_0}v^{\epsilon}_{a_0}-\hat{\omega}^{\alpha\epsilon}_{a_0}\Big)r_{a\epsilon}-a_{a_0\epsilon}\Big(r^{\alpha}_a r^{\epsilon}_a-\tfrac{1}{2}\gamma^{\alpha\epsilon}r_{a\lambda}r^{\lambda}_a\Big)+\mathcal{O}(c^{-2}), \tag{349}$$

were the anti-symmetric relativistic precession matrix $\hat{\omega}^{\alpha\beta}_{a_0}$ in the barycentric reference frame is given as below:

$$\dot{\hat{\omega}}^{\alpha\beta}_{a_0} = \tfrac{3}{2}(v^{\alpha}_{a_0}a^{\beta}_{a_0}-v^{\beta}_{a_0}a^{\alpha}_{a_0})-2\sum_{b\neq a}\Big(\frac{\partial\bar{w}^{\beta}_b}{\partial x_\alpha}-\frac{\partial\bar{w}^{\alpha}_b}{\partial x_\beta}\Big)+\mathcal{O}(c^{-2}). \tag{350}$$

Also, the function $\hat{\mathcal{L}}_a$ was determined in the following form:

$$\hat{\mathcal{L}}_a(x) = \hat{\ell}_{a_0}+\hat{\ell}_{a_0\lambda}r^{\lambda}_a-\tfrac{1}{2}c\,r_{a\lambda}r_{a\mu}\Big[v^{\lambda}_{a_0}a^{\mu}_{a_0}+v^{\mu}_{a_0}a^{\lambda}_{a_0}+\gamma^{\lambda\mu}\sum_{b\neq a}\dot{\bar{U}}_b+2\sum_{b\neq a}\Big(\frac{\partial\bar{w}^{\mu}_b}{\partial x_\lambda}+\frac{\partial\bar{w}^{\lambda}_b}{\partial x_\mu}\Big)\Big]-$$

$$-\tfrac{1}{10}c(\dot a_{a_0\epsilon}r^\epsilon_a)(r_{a\nu}r^\nu_a)+\delta\hat\ell_a(x)+\mathcal{O}(c^{-2}), \tag{351}$$

with functions $\hat\ell_{a_0}$ and $\hat\ell^\alpha_{a_0}$ are given by

$$\begin{aligned}\tfrac{1}{c}\dot{\hat\ell}_{a_0} &= -\tfrac{1}{8}(v_{a_0\epsilon}v^\epsilon_{a_0})^2+\tfrac{3}{2}(v_{a_0\epsilon}v^\epsilon_{a_0})\sum_{b\neq a}\bar U_b-4v_{a_0\epsilon}\sum_{b\neq a}\bar w^\epsilon_b+\tfrac{1}{2}\big(\sum_{b\neq a}\bar U_b\big)^2-\sum_{b\neq a}\delta\bar w_b+\mathcal{O}(c^{-2}), && (352)\\ \tfrac{1}{c}\hat\ell^\alpha_{a_0} &= \dot{\hat q}^\alpha_{a_0}-\tfrac{1}{2}v^\alpha_{a_0}(v_{a_0\epsilon}v^\epsilon_{a_0})+3v^\alpha_{a_0}\sum_{b\neq a}\bar U_b-4\sum_{b\neq a}\bar w^\alpha_b+\mathcal{O}(c^{-2}). && (353)\end{aligned}$$

Finally, the function $\hat q^\alpha_{a_0}$ is the post-Newtonian part of the position vector of the body a given by Eq. (344). Together with the Newtonian part of the acceleration, $\ddot z_{a_0}=a^\alpha_{a_0}$, the post-Newtonian part $\ddot{\hat q}^\alpha_{a_0}$ completes the barycentric equation of motions of the body a given by Eq. (345). This equation essentially is the geodetic equation of motion written for the body a in the global reference frame.

Substituting these solutions for the functions $\hat{\mathcal{K}}_a$, $\hat{\mathcal{Q}}^\alpha_a$ and $\hat{\mathcal{L}}_a$ into the expressions for the potentials w and w^α given by Eqs. (282)–(283), we find the following form for these potentials:

$$\begin{aligned} w(x) &= \sum_b w_b(x)-\sum_{b\neq a}\Big(\bar w_b+r^\epsilon_a\frac{\partial\bar w_b}{\partial x^\epsilon}\Big)-\\ &\quad -\frac{1}{c^2}\Big\{\tfrac{1}{2}r^\epsilon_a r^\lambda_a\Big[3a_{a_0\epsilon}a_{a_0\lambda}+\dot a_{a_0\epsilon}v_{a_0\lambda}+v_{a_0\epsilon}\dot a_{a_0\lambda}+\gamma_{\epsilon\lambda}\sum_{b\neq a}\ddot{\bar U}_b+2\sum_{b\neq a}\Big(\frac{\partial}{\partial x^\epsilon}\dot{\bar w}_{b\lambda}+\frac{\partial}{\partial x^\lambda}\dot{\bar w}_{b\epsilon}\Big)\Big]+\\ &\quad +\tfrac{1}{10}(\ddot a_{a_0\epsilon}r^\epsilon_a)(r_{a\mu}r^\mu_a)-\Big(c\frac{\partial}{\partial x^0}+v^\epsilon_{a_0}\frac{\partial}{\partial x^\epsilon}\Big)\tfrac{1}{c}\delta\hat\ell_a\Big\}+\mathcal{O}(c^{-4}), && (354)\\ w^\alpha(x) &= \sum_b w^\alpha_b(x)-\sum_{b\neq a}\Big(\bar w^\alpha_b+r^\epsilon_a\frac{\partial\bar w^\alpha_b}{\partial x^\epsilon}\Big)-\tfrac{1}{10}\big\{3r^\alpha_a r^\epsilon_a-\gamma^{\alpha\epsilon}r_{a\mu}r^\mu_a\big\}\dot a_{a_0\epsilon}+\tfrac{1}{4c}\frac{\partial}{\partial x_\alpha}\delta\hat\ell_a+\mathcal{O}(c^{-2}). && (355)\end{aligned}$$

The same argument that allowed us to eliminate $\delta\ell_a$ in Sec. IV C works here, allowing us to omit $\delta\hat\ell_a$. Therefore, we can now present the metric of the moving observer expressed in the global coordinates $\{x^n\}$. Substituting the expressions for the potentials given by (354) and (355) into Eqs. (279)–(281) we obtain the following expressions

$$\begin{aligned} g^{00}(x) &= 1+\frac{2}{c^2}\Big\{\sum_b w_b(x)-\sum_{b\neq a}\Big(\bar w_b+r^\epsilon_a\frac{\partial\bar w_b}{\partial x^\epsilon}\Big)\Big\}+\frac{2}{c^4}\Big\{\Big[\sum_b U_b(x)-\sum_{b\neq a}\Big(\bar U_b+r^\epsilon_a\frac{\partial\bar U_b}{\partial x^\epsilon}\Big)\Big]^2-\\ &\quad -\tfrac{1}{2}r^\epsilon_a r^\lambda_a\Big(3a_{a_0\epsilon}a_{a_0\lambda}+\dot a_{a_0\epsilon}v_{a_0\lambda}+v_{a_0\epsilon}\dot a_{a_0\lambda}+\gamma_{\epsilon\lambda}\sum_{b\neq a}\ddot{\bar U}_b+2\sum_{b\neq a}(\partial_\epsilon\dot{\bar w}_{b\lambda}+\partial_\lambda\dot{\bar w}_{b\epsilon})\Big)-\\ &\quad -\tfrac{1}{10}(\ddot a_{a_0\epsilon}r^\epsilon_a)(r_{a\mu}r^\mu_a)\Big\}+\mathcal{O}(c^{-6}), && (356)\\ g^{0\alpha}(x) &= \frac{4}{c^3}\Big\{\sum_b w^\alpha_b(x)-\sum_{b\neq a}\Big(\bar w^\alpha_b+r^\epsilon_a\frac{\partial\bar w^\alpha_b}{\partial x^\epsilon}\Big)-\tfrac{1}{10}\big\{3r^\alpha_a r^\epsilon_a-\gamma^{\alpha\epsilon}r_{a\mu}r^\mu_a\big\}\dot a_{a_0\epsilon}\Big\}+\mathcal{O}(c^{-5}), && (357)\\ g^{\alpha\beta}(x) &= \gamma^{\alpha\beta}-\gamma^{\alpha\beta}\frac{2}{c^2}\Big\{\sum_b w_b(x)-\sum_{b\neq a}\Big(\bar w_b+r^\epsilon_a\frac{\partial\bar w_b}{\partial x^\epsilon}\Big)\Big\}+\mathcal{O}(c^{-4}). && (358)\end{aligned}$$

The coordinate transformations that put the observer in this reference frame are given by:

$$\begin{aligned} y^0_a &= x^0+c^{-2}\Big\{\int_{x^0_0}^{x^0}\Big(\tfrac{1}{2}\big(v_{a_0\epsilon}+c^{-2}\dot{\hat q}_{a_0\epsilon}\big)\big(v^\epsilon_{a_0}+c^{-2}\dot{\hat q}^\epsilon_{a_0}\big)-\sum_{b\neq a}\bar w_b\Big)dx'^0+\\ &\quad +c\,\Big(\big(v_{a_0\epsilon}+c^{-2}\dot{\hat q}_{a_0\epsilon}\big)\big(1+c^{-2}\big(3\sum_{b\neq a}\bar U_b-\tfrac{1}{2}v_{a_0\mu}v^\mu_{a_0}\big)-c^{-2}4\sum_{b\neq a}\bar w_{b\epsilon}\Big)r^\epsilon_a\Big\}+\\ &\quad +c^{-4}\Big\{\int_{x^0_0}^{x^0}\Big(-\tfrac{1}{8}(v_{a_0\epsilon}v^\epsilon_{a_0})^2+\tfrac{3}{2}(v_{a_0\epsilon}v^\epsilon_{a_0})\sum_{b\neq a}\bar U_b-4v_{a_0\epsilon}\sum_{b\neq a}\bar w^\epsilon_b+\tfrac{1}{2}\big(\sum_{b\neq a}\bar U_b\big)^2\Big)dx'^0+\\ &\quad -\tfrac{1}{2}c\,r_{a\lambda}r_{a\mu}\Big(v^\lambda_{a_0}a^\mu_{a_0}+v^\mu_{a_0}a^\lambda_{a_0}+\gamma^{\lambda\mu}\sum_{b\neq a}\dot{\bar U}_b+2\sum_{b\neq a}(\partial^\lambda\bar w^\mu_b+\partial^\mu\bar w^\lambda_b)\Big)-\tfrac{1}{10}c(\dot a_{a_0\epsilon}r^\epsilon_a)(r_{a\nu}r^\nu_a)\Big\}+\mathcal{O}(c^{-6}), && (359)\end{aligned}$$

$$y_a^{\alpha} \;=\; r_a^{\alpha} - c^{-2}\Big\{\Big(\tfrac{1}{2}v_{a_0}^{\alpha}v_{a_0}^{\epsilon} + \hat{\omega}_{a_0}^{\alpha\epsilon} - \gamma^{\alpha\epsilon}\sum_{b\neq a}\bar{U}_b\Big)r_{a\epsilon} + a_{a_0\epsilon}\Big(r_a^{\alpha}r_a^{\epsilon} - \tfrac{1}{2}\gamma^{\alpha\epsilon}r_{a\lambda}r_a^{\lambda}\Big)\Big\} + \mathcal{O}(c^{-4}). \tag{360}$$

The obtained inverse coordinate transformations are new and extend the previously obtained results (for review, see [5, 7], and also [6]). This set of results concludes our derivation of the coordinate transformations from a local accelerated reference from to the global inertial frame. Generalization to the case of N extended bodies can be done in a complete analogy with Eqs. (258)-(260), which is sufficient for the solar system applications.

VI. DISCUSSION AND CONCLUSIONS

In this paper we introduced a new approach to find a solution for the N-body problem in the general theory of relativity in the weak-field and slow motion approximation. The approach is based on solving all three problems discussed in Sec. I, including finding solutions to global and local problems, as well as developing a theory of relativistic reference frames. Our objective was to establish the properties of a local frame associated with the world-line of one of the bodies in the system and to find a suitable form of the metric tensor corresponding to this frame; we achieved this goal by imposing a set of clearly defined coordinate and physical conditions.

Specifically, we combined a new perturbation theory ansatz introduced in Sec. II C for the gravitational N-body system. In this approach, the solution to the gravitational field equations in any reference frame is presented as a sum of three terms: i) the inertial flat spacetime in that frame, ii) unperturbed solutions for each body in the system boosted to the coordinates of this frame, and iii) the gravitational interaction term. We use the harmonic gauge conditions that allow reconstruction of a significant part of the structure of the post-Galilean coordinate transformation functions relating non-rotating global coordinates in the inertial reference frame to the local coordinates of the non-inertial frame associated with a particular body. The remaining parts of these functions are determined from dynamical conditions, obtained by constructing the relativistic proper reference frame associated with a particular body. In this frame, the effect of external forces acting on the body is balanced by the fictitious frame-reaction force that is needed to keep the body at rest with respect to the frame, conserving its relativistic linear momentum. We find that this is sufficient to determine explicitly all the terms of the coordinate transformation. Exactly the same method is then used to develop the inverse transformations.

Our approach naturally incorporates properties of dynamical coordinate reference systems into the hierarchy of relativistic reference systems in the solar system and relevant time scales, accurate to the c^{-4} order. The results obtained enable us to address the needs of practical astronomy and allow us to develop adequate models for high-precision experiments. In our approach, an astronomical reference system is fully defined by a metric tensor given in a particular coordinate frame, a set of gravitational potentials describing the gravitational field in that frame, a set of direct, inverse, and mutual coordinate transformations between the frames involved, relevant time arguments, ephemerides, and the standard physical constants and algorithms. We anticipate that the results presented here may find immediate use in many areas of modern geodesy, astronomy, and astrophysics.

The new results reported in this paper agree well with those previously obtained by other researchers. In fact, the coordinate transformations that we derived here are in agreement with the results established for both direct and inverse coordinate transformations given in Refs [11–14] and [5, 9, 10, 15], correspondingly. However, our approach allows one to consistently and within the same framework develop both direct and inverse transformations, the corresponding equations of motion, and explicit forms of the metric tensors and gravitational potentials in the various reference frames involved. The difficulty of this task was mentioned in Ref. [6] when the post-Newtonian motion of a gravitational N-body system was considered; our proposed formulation successfully resolves this important issue. As an added benefit, the new approach provides one with a good justification to eliminate the functions $\delta\kappa$, $\delta\xi$ and $\delta\ell$, yielding a complete form for the transformation functions $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{Q}^{\alpha}$ involved in the transformations (as well as their "hatted" inverse counterparts).

The significance of our result is that for the first time, a formalism for the coordinate transformation between relativistic reference frames is provided, presenting both the direct and inverse transformations in explicit form. By combining inverse and direct transformations, the transformation rules between arbitrary accelerating frames can be obtained. Furthermore, it is possible to combine direct (or inverse) transformations, and obtain a complete set of transformations that can be represented by our formalism, as shown explicitly in [8, 16]. This leads to an *approximate* finite group structure that extends the Poincaré group of global transformations to accelerating reference frames.

The results obtained in this paper are designed to facilitate the analysis of relativistic phenomena with ever increasing greater accuracy. We should note that the approach we presented can be further developed in an iterative manner: if greater accuracy is desired, the coordinate transformations (37)–(38) can be expanded to include higher-order terms. Furthermore, the same approach relying on the functional $\mathcal{KLQ}$-parameterization may be successfully applied to the case of describing the gravitational dynamics of an astronomical N-body system and dynamically rotating reference

frames. This work has begun and the results, when available, will be reported elsewhere.

There are some problems that remain to be solved. One of these is the problem of relativistic rotation. In particular, it is known that the rotational motion of extended bodies in general relativity is a complicated problem that has no complete solution up to now. On the other hand, modern observational accuracy of the geodynamical observations makes it necessary to have a rigorous relativistic model of the Earth's rotation. The method presented in this paper may now be used to study the rotation of extended bodies to derive a theory of relativistic rotation within the general theory of relativity. Our $\mathcal{KLQ}$ formalism presented here provides one with the necessary conceptual basis to study this problem from a very general position and could serve as the foundation for future work. Also, a complete formal treatment of the case of a system of N extended bodies should be presented. Although this treatment would result in corrections much smaller to those anticipated in the solar system experiments, they may be important to a stronger gravitational regime, such as in binary pulsars. This work had been initiated. The results will be reported elsewhere.

Our current efforts are directed towards the practical application of the results obtained in this paper (similar to those discussed in [26] in the context of the GRAIL mission). We aim to establish the necessary relativistic measurements models, equations of motion for the celestial bodies and spacecraft, as well as the light propagation equations in different frames involved; we also plan to implement these results in the form of computer codes for the purpose of high-precision spacecraft navigation and will perform the relevant scientific data analysis. The analysis of the above-mentioned problems from the standpoint of the new theory of relativistic astronomical reference systems will be the subject of specific studies and future publications.

Acknowledgments

We thank Sami W. Asmar, William M. Folkner, Michael M. Watkins, and James G. Williams of JPL for their interest, support and encouragement during the work and preparation of this manuscript. It is our pleasure to acknowledge fruitful conversations with Olivier L. Minazzolli. This work was performed at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration.

---

[1] C. M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, UK, 1993).
[2] S. G. Turyshev, Usp. Fiz. Nauk **179**, 3 (2009), [Phys. Usp. **52**, 1-27 (2009)], arXiv:0809.3730 [gr-qc].
[3] C. M. Will, Living Rev. Relativity **9** (2006), gr-qc/0510072, URL `http://www.livingreviews.org/lrr-2006-3`.
[4] S. G. Turyshev, U. E. Israelsson, M. Shao, N. Yu, A. Kusenko, et al., Int.J.Mod.Phys. **D16**, 1879 (2008), arXiv:0711.0150.
[5] V. A. Brumberg and S. M. Kopeikin, in *Reference Frames*, edited by J. Kovalevsky, I. I. Mueller, & B. Kolaczek (1989), vol. 154 of *Astrophysics and Space Science Library*, pp. 115–141.
[6] M. Soffel, S. A. Klioner, G. Petit, P. Wolf, S. M. Kopeikin, P. Bretagnon, V. A. Brumberg, N. Capitaine, T. Damour, T. Fukushima, et al., Astrophys. J. **126**, 2687 (2003), astro-ph/0303376.
[7] V. A. Brumberg, *Essential Relativistic Celestial Mechanics* (Hilger, Bristol, 1991).
[8] S. G. Turyshev, Tech. Rep., NASA Jet Propulsion Laboratory, Publication #96-13 (1996), arXiv:gr-qc/9606063.
[9] S. M. Kopeikin, Celestial Mechanics **44**, 87 (1988).
[10] V. A. Brumberg and S. M. Kopejkin, Nuovo Cimento B Series **103**, 63 (1989).
[11] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **43**, 3273 (1991).
[12] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **45**, 1017 (1992).
[13] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **47**, 3124 (1993).
[14] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **49**, 618 (1994).
[15] S. M. Kopeikin and I. Vlasov, Phys.Rept. **400**, 209 (2004), gr-qc/0403068.
[16] S. G. Turyshev, O. L. Minazzoli, and V. T. Toth, J.Math.Phys. **53**, 032501 (2012), arXiv:1109.1796.
[17] V. A. Fock, *The Theory of Space, Time and Gravitation* (Fizmatgiz, Moscow (in Russian), 1959), [English translation (1959), Pergamon, Oxford].
[18] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman and Co., 1973).
[19] W. M. Yao et al. (Particle Data Group), J. Phys. G **33**, 205 (2006).
[20] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields.* (7-th edition. Nauka: Moscow (in Russian), 1988).
[21] A. Einstein, Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin pp. 844–847 (1915).
[22] A. Einstein, Annalen der Physik **49**, 146 (1916).
[23] S. Chandrasekhar, Astrophys. J. **142**, 1488 (1965).
[24] K. S. Thorne, Rev. Mod. Phys. **52**, 299 (1980).
[25] V. I. Denisov and V. G. Turyshev, Theor. Mat. Phys. **83**, 429 (1990), (in Russian).
[26] S. G. Turyshev, V. T. Toth, and M. V. Sazhin, Phys. Rev. D **87**, 024020 (2013), arXiv:1212.0232.